THE APRIL MEETING OF THE CHICAGO SECTION.

THE third regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Saturday, April 9th, the first session opening at 10 o'clock a. m. The following members were present:

Dr. E. M. Blake, Dr. Harris Hancock, Professor Thomas F. Holgate, Mr. Herbert G. Keppel, Dr. Kurt Laves, Professor Heinrich Maschke, Professor E. H. Moore, Professor J. B. Shaw, Dr. H. F. Stecker, Professor H. S. White, Professor Alexander Ziwet.

Professor E. H. Moore, Vice-President of the Society, occupied the chair.

The following papers were read:

(1) Professor JAMES BYRNIE SHAW: "Dual algebras."

(2) Dr. E. M. BLAKE: "The ellipsograph of Proclus and its inverse."

This paper, which embodied an earlier one* upon the same mechanism, described, first, the surfaces generated by any straight line carried by either of two circles, of radius r and $2r$ respectively, lying in the same plane and rolling the one upon the other, these surfaces being of the fourth order; and second, the envelopes of carried planes, these being of the sixth order and fourth class for the ellipsograph, and cones for the inverse mechanism. The paper was illustrated by thread models showing the principal types of these surfaces.

(3) Professor ROBERT J. ALEY: "A triangle related to Nagel's triangle."

(4) Dr. L. E. DICKSON: "The structure of the hypo-abelian groups."

The paper announced marked simplifications in the general conception and in the detailed developments of the theory of the hypo-abelian groups of Jordan, and of the author's generalization of the first hypo-abelian group just appearing in the *Quarterly Journal of Mathematics*.

(5) Dr. KURT LAVES: "On the most general form of the inner potential consistent with the complete integration of the differential equations of motion of a free system of two bodies" (preliminary communication).

*"On a ruled surface, etc.," abstract in BULLETIN, vol. IV., pp. 186. This paper will not be published separately, owing to its incorporation in the present one.

(6) Dr. SHUNKICHI KIMURA (presented by Professor J. B. Shaw) :

I. "Quaternion notes."

II. "Introduction to the theory of functions of a quaternion or a vector variable."

(7) Professor W. H. METZLER : "On the roots of a determinantal equation."

In the *American Journal of Mathematics*, vol. 19, Dr. Thomas Muir showed that the roots of the equation

$$\begin{vmatrix} 11-x, & 21 & , 13 & , \cdots \\ 21 & , 22-x, & 23 & , \cdots \\ 31 & , 32 & , 33-x, & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{vmatrix} = 0$$

are all real in case $1\mu \cdot \mu\nu \cdot \nu 1 = \mu 1 \cdot \nu\mu \cdot 1\nu$ and $1\mu \cdot \mu 1$ is positive. In the present paper the writer shows that the roots of this determinantal equation are pure imaginaries when $1\mu \cdot \mu\nu \cdot \nu 1 = -\mu 1 \cdot \nu\mu \cdot 1\nu$, $1\mu \cdot \mu 1$ is negative, and $\lambda\lambda = 0$.

(8) Professor H. MASCHKE : "Concerning the case where a linear substitution-group of finite order in n variables breaks up into groups in a lower number of variables."

If a linear substitution-group G in n variables is of finite order and contains at least one substitution, the characteristic equation of which has all its roots distinct, then the following theorem holds true : If at least one coefficient occupying a certain place (not in the main diagonal) in the matrix of the substitutions of G vanishes for *all* the substitutions, then the group breaks up into two groups in $n - p$ and in p variables, respectively. The paper containing the proof of this theorem will be published in the *Mathematische Annalen*, vol. 50.

(9) Professor E. H. MOORE : "A two-parameter class of solvable quintics, in which the rational relations amongst the roots, by threes, do not contain the parameters" (preliminary communication).

At the opening of the afternoon session Professor Michelson, of the University of Chicago, favored the section with an exhibition of the workings of his new harmonic analyser, a description of which was published in the *American Journal of Science* and in the *Philosophical Magazine* for January, 1898.

THOMAS F. HOLGATE,
Secretary of the Section.

THE THEOREMS OF OSCILLATION OF STURM AND KLEIN. (SECOND PAPER.)

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

THE following pages form a continuation of §§ 1, 2 of a paper presented under the same title to the Society at its December meeting and printed on pp. 295–313 of the present volume of the BULLETIN. This paper will be referred to for brevity as *Th. of Osc. 1.* No use is here made of the results contained in § 3 of the paper just mentioned, these results being merely very special cases of some of those now given.

In § 1 of the present paper, after proving two simple theorems of Sturm, by a method different from that used by him, I have used them to throw Sturm's theorem of oscillation into a slightly generalized form. In § 2 I have proved Klein's theorem of oscillation in a very general form, although as I have restricted myself here, as in the previous paper, to the case in which the coefficients of the binomial differential equations are continuous the cases here considered are not yet the most general ones. I expect in a subsequent paper to come back to these more general cases which do not seem to present any serious difficulty.

The truth of the theorems proved in § 2 (at least in important special cases) has been suspected by Klein partly from analogy with the special cases in which the differential equation has a polynomial solution, and partly from analogy with the simple cases of Lamé's equations (I use the term here in Klein's most general sense) with two or at most three variable parameters, in which cases he had given rough geometrical proofs which however made no pretense at rigor.* In the more complicated cases no proof of any sort has as far as I know been given.

§ 1. *Sturm's Generalized Theorem of Oscillation.*

We will assume that in the interval $a \leqq x \leqq b$ $p(x)$ and $q(x)$ are single valued continuous real functions of the real

*Cf. Klein's lithographed lectures: "Ueber lineare Differentialgleichungen der zweiten Ordnung," p. 401–431. We shall see that the restriction which Klein here makes to equations which are everywhere regular is unessential.

variable x. We know (*Th. of Osc. 1*, (E)) that in this interval no solution of the equation :

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{1}$$

which is not identically zero can have more than a finite number of roots ; and also* that, if y_1 and y_2 are two linearly independent solutions, between two successive roots of y_1 lies one and only one root of y_2.

We proceed now to a theorem which enables us to compare different solutions of the same differential equation in the same way as the first theorem of comparison (*Th. of Osc. 1*, Theorem I) enabled us to compare corresponding solutions of different equations :

I. *Assuming that y_1 and y_2 are two solutions of* (1) *such that $y_2(a) \neq 0$ and (except when $y_1(a) = 0$ when no further assumption is necessary) $y_1'(a)/y_1(a) > y_2'(a)/y_2(a)$; then, if y_1 has n roots x_i such that $a < x_1 < x_2 \cdots < x_n \leqq b$, y_2 has just n roots in the interval $a < x < x_n$.*

For in the first place, since y_2 is clearly linearly independent of y_1, in each of the intervals $x_i \leqq x \leqq x_{i+1}$ $(i = 1, 2, \cdots, n-1)$ y_2 has one and only one root. It is then merely necessary to prove that the same is true for the interval $a < x \leqq x_1$. If $y_1(a) = 0$ this follows as before since a and x_1 are then two successive roots of y_1. If $y_1(a) \neq 0$ it is also clear at once that y_2 cannot have more than one root in the interval $a < x \leqq x_1$, as otherwise y_1 would have a root there. If then we can prove that the assumption that y_2 has no root in the interval just mentioned leads to a contradiction, our theorem will be proved. Consider the function :

$$\bar{y}(x) = \frac{y_1(x)}{y_1(a)} - \frac{y_2(x)}{y_2(a)}$$

which is clearly a solution of (1) linearly independent of y_1. Since $\bar{y}(a) = 0$ and $\bar{y}'(a) > 0$, $\bar{y}(a+\varepsilon)$ will, for small positive values of ε, be positive. Now if $y_2(x)$ had no root in the interval $a < x \leqq x_1$ $y_2(x_1)$ and $y_2(a)$ would have the same signs, so that $\bar{y}(x_1) = -y_2(x_1)/y_2(a) < 0$, and therefore $\bar{y}(x)$ would have at least one root in the interval $a < x < x_1$. Therefore, since $\bar{y}(a) = 0$, $y_2(x)$ would have at least one root in this interval, and this is contrary to hypothesis.

*I have recently had occasion to discuss this familiar theorem of Sturm. See BULLETIN for March, 1897, p. 210.

We come now to a proposition analogous to the second theorem of comparison (*Th. of Osc. 1*, Theorem II):

II. *Assuming that y_1 and y_2 are two solutions of* (1) *satisfying the same conditions as in Theorem I, and having in the interval $a < x < b$ the same number n of roots (n may be zero); then if $y_2(b) + 0$ we shall have $y_1(b) + 0$ and*

$$\frac{y_1'(b)}{y_1(b)} > \frac{y_2'(b)}{y_2(b)}.$$

That $y_1(b) + 0$ follows at once from Theorem I, as otherwise y_2 would have $n + 1$ roots in the interval $a < x < b$. In order to prove the remainder of the theorem we will use Abel's formula:

$$y_2y_1' - y_1y_2' = ke^{-\int p dx}$$

in which k is a real constant different from zero. Dividing this equation through by y_1y_2 we see that, except when y_1 or y_2 is zero, $y_1'/y_1 - y_2'/y_2$ has the same sign as k/y_1y_2. Moreover this sign is by hypothesis positive when $x = a$, except when $y_1(a) = 0$, in which case it is clear that for a value of x a little greater than a the sign is positive since it is the same as the sign of y_1'/y_1. Now when $x = b$ the sign of y_1y_2 is the same as when x is equal to or a little greater than a, since y_1 and y_2 change sign the same number of times in the interval $a < x < b$. Accordingly $y_1'/y_1 - y_2'/y_2$ is positive when $x = b$, as was to be proved.

We proceed now to a generalized form of Sturm's theorem of oscillation:

III. *If when $a \leqq x \leqq b$ and $L > \lambda > l$*, $\varphi(x, \lambda)$ is a single valued continuous function of (x, λ) and if, when $a \leqq x \leqq b$ and $L > \lambda' > \lambda'' > l$, $\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$†; if moreover*

$$\lim_{\lambda=L} \varphi(x, \lambda) = +\infty \text{ and } \lim_{\lambda=l} \varphi(x, \lambda) = -\infty$$

EXCEPT PERHAPS FOR A FINITE NUMBER OF VALUES OF x IN THE INTERVAL $a \leqq x \leqq b$; *then there will be one and only one value of λ ($L > \lambda > l$) for which the equation:*

$$\frac{d^2y}{dx^2} = \varphi(x, \lambda)\, y \tag{2}$$

* We may have here $L = +\infty$, or $l = -\infty$, or both L and l may be infinite.

† The equality sign must not hold for all values of x in the interval $a \leqq x \leqq b$ (cf. *Th. of Osc. 1*, p. 304 and 305, footnotes).

has a solution which has just m roots in the interval $a < x < b$ and which satisfies the conditions:

$$\alpha' y(a) - \alpha y'(a) = 0,$$
$$\beta' y(b) - \beta y'(b) = 0.$$

It should be noted that except for the clause printed in small capitals the theorem just stated is identical in substance (although stated in a somewhat different form) with the simple form of Sturm's theorem of oscillation (*Th. of Osc. 1*, Theorem VIII).* In order to prove the theorem in all its generality it is clearly sufficient to prove that: (α) we can find a quantity l_1 such that when $l < \lambda \leqq l_1$ every solution of (2) has more than m roots in the interval $a < x < b$; (β) we can find a quantity L_1 such that when $L_1 \leqq \lambda < L$ the solution of the differential equation for which $y(a) = \alpha$, $y'(a) = \alpha'$ does not have any root in the interval $a < x \leqq b$ and (when $\beta \neq 0$) such that $y'(b)/y(b) > \beta'/\beta$.

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* This identity follows from the following fact:

If when $a \leqq x \leqq b$ and $L > \lambda > l$, $\phi(x, \lambda)$ is a continuous function of (x, λ); and if when $L > \lambda' > \lambda'' > l$, $\phi(x, \lambda') \geqq \phi(x, \lambda'')$; and if for every value of x in the interval $a \leqq x \leqq b$ $\lim_{\lambda = L} \phi(x, \lambda) = +\infty$; then no matter how large a positive quantity M we may choose it is possible to find a quantity K such that when $L > \lambda > K$ and $a \leqq x \leqq b$, $\phi(x, \lambda) > M$.

The idea contained in this theorem might be expressed by saying that $\phi(x, \lambda)$ diverges towards $+\infty$ *uniformly* throughout the interval $a \leqq x \leqq b$. As I am unable to refer to any place where this theorem is proved I will give a proof of it here.

Take an infinite number of values $\lambda', \lambda'', \lambda''', \ldots$ such that $l < \lambda' < \lambda'' < \ldots < L$ and such that $\lim_{n=\infty} \lambda^{[n]} = L$. Then we have $\phi(x, \lambda') < \phi(x, \lambda'') < \ldots$ and $\lim_{n=\infty} \phi(x, \lambda^{[n]}) = +\infty$. Now let x_n be that value (or one of the values) of x in the interval $a \leqq x \leqq b$ at which $\phi(x, \lambda^{[n]})$ is smallest. Then either $\lim_{n=\infty} \phi(x_n, \lambda^{[n]}) = +\infty$, in which case our theorem is evidently true, or we can find a positive quantity M so large that no matter how large n is $\phi(x_n, \lambda^{[n]}) < M$. To show that this last case is impossible let us consider the points $x_1, x_2, x_3, \ldots$. These points must have at least one limiting point in the interval $a \leqq x \leqq b$. Let $\bar{x}$ be such a limiting point. Now by hypothesis it is possible to take λ so near to L that $\phi(\bar{x}, \lambda) > M$. Suppose then that when $n > n_1$, $\phi(\bar{x}, \lambda^{[n]}) > M$. Since $\phi(x, \lambda^{[n]})$ is continuous at $\bar{x}$ it is possible to take the positive quantity ε so small that throughout the interval from $\bar{x} - \varepsilon$ to $\bar{x} + \varepsilon$ $\phi(x, \lambda^{[n_1]}) > M$, and accordingly when $n > n_1$, $\phi(x, \lambda^{[n]}) > M$ throughout this interval. But this is impossible since there are within this interval an infinite number of the points $x_1, x_2, \ldots$ at which $\phi(x_n, \lambda^{[n]}) < M$.

I will add that from this theorem may be deduced (or conversely that this theorem may be deduced from) the theorem that if a series whose terms are continuous functions converges throughout the interval $a \leqq x \leqq b$ to a continuous limit and if the terms of the series in this interval are nowhere negative, the series is uniformly convergent.

That we can find a quantity l_1 as described in (α) is obvious at once. For let a_1 and b_1 be two points, such that $a \leqq a_1 < b_1 \leqq b$ but such that none of the points at which the formula $\lim_{\lambda=l} \varphi(x,\lambda) = -\infty$ fails to hold lie in the interval $a_1 \leqq x \leqq b_1$. Then we can take l_1 so that when $l < \lambda \leqq l_1$ $\varphi(x,\lambda) < -\frac{(m+1)^2\pi^2}{(b-a)^2}$ $(a_1 \leqq x \leqq b_1)$ so that (*Th. of Osc. 1*, Theorem V) every solution of (2) has at least $m+1$ roots in the interval $a_1 < x < b_1$ and therefore *a fortiori* in the interval $a < x < b$.

The proof that we can find a quantity L_1 as described in (β) is less simple. Let us first consider the case in which the formula $\lim_{\lambda=L} \varphi(x,\lambda) = +\infty$ holds at all points of the interval $a \leqq x \leqq b$ except at one or both of the extremities a, b.* Let us denote by $y_1(x,\lambda)$ and $y_2(x,\lambda)$ the solutions of (2) which satisfy respectively the relations:

$$y_1(a,\lambda) = \alpha,\ y_1'(a,\lambda) = \alpha' \text{ and } y_2(b,\lambda) = \beta,\ y_2'(b,\lambda) = \beta'.$$

Choose arbitrarily a fixed quantity $\bar{\lambda}$ such that $l < \bar{\lambda} < L$. Then since $y_1(x,\bar{\lambda})$ and $y_2(x,\bar{\lambda})$ each has at most a finite number of roots in the interval ab it is possible to find a positive quantity ε so small that letting $a + \varepsilon = a_1$ and $b - \varepsilon = b_1$ $y_1(x,\bar{\lambda})$ has no root in the interval $a < x \leqq a_1$ and $y_2(x,\bar{\lambda})$ has no root in the interval $b_1 \leqq x < b$. It follows by the first theorem of comparison (*Th. of Osc. 1*, Theorem I) that when $\bar{\lambda} \leqq \lambda < L$, $y_1(x,\lambda)$ and $y_2(x,\lambda)$ have no roots in the intervals $a < x \leqq a_1$ and $b_1 \leqq x < b$ respectively. By the second theorem of comparison (*Th. of Osc. 1*, Theorem II) it follows that when $\bar{\lambda} < \lambda < L$:

$$\frac{y_1'(a_1,\lambda)}{y_1(a_1,\lambda)} > \frac{y_1'(a_1,\bar{\lambda})}{y_1(a_1,\bar{\lambda})} \text{ and } \frac{y_2'(b_1,\lambda)}{y_2(b_1,\lambda)} < \frac{y_2'(b_1,\bar{\lambda})}{y_2(b_1,\bar{\lambda})}\dagger$$

If now we denote by $\bar{y}(x,\lambda)$ the solution of (2) for which $\bar{y}(a_1,\lambda) = y_1(a_1,\bar{\lambda})$ and $\bar{y}'(a_1,\lambda) = y_1'(a_1,\bar{\lambda})$ (so that when $\lambda = \bar{\lambda}$, y_1 and $\bar{y}$ are identical), we see (*Th. of Osc. 1*, Theorems III and IV) that it is possible to find a quantity

* This is, in fact, the only case we shall make any application of in this paper. The more general case would, however, be necessary in the problem mentioned in the third footnote on p. 376.

† The reversal of the sign of inequality here is due to the fact that in order to apply the theorem of comparison we must introduce the new variable $x' = -x$ and this reverses the sign of y_2'.

L_1 ($\bar{\lambda} < L_1 < L$) such that when $L_1 \leqq \lambda < L$, $\bar{y}(x, \lambda)$ has no roots in the interval $a_1 \leqq x \leqq b_1$ and $\bar{y}'(b_1, \lambda) / \bar{y}(b_1, \lambda) > y_2'(b_1, \bar{\lambda}) / y_2(b_1, \bar{\lambda})$. Let us now compare y_1 and y with each other by applying Theorems I and II of the present paper to the interval $a_1 \leqq x \leqq b_1$. We see that when $L_1 < \lambda < L$, $y_1(x, \lambda)$ has no roots in the interval $a_1 \leqq x \leqq b_1$ (and therefore none in the interval $a < x \leqq b_1$) and that $y_1'(b_1, \lambda) / y_1(b_1, \lambda) > y_2'(b_1, \bar{\lambda}) / y_2(b_1, \bar{\lambda}) > y_2'(b_1, \lambda) / y_2(b_1, \lambda)$. Finally applying the same theorems again to the interval $b_1 \leqq x \leqq b$ so as to compare y_1 and y_2 we see that when $L_1 \leqq \lambda < L$, $y_1(x, \lambda)$ has no roots in this interval (and therefore none in the interval $a < x < b$ as was to be proved), and that: $y_1'(b, \lambda) / y_1(b, \lambda) > y_2'(b, \lambda) / y_2(b, \lambda) = \beta' / \beta$ (as was also to be proved).

If now the formula $\lim_{\lambda = L} \varphi(x, \lambda) = +\infty$ fails to hold at the points $c_1, c_2, \cdots, c_k$, as well as perhaps at the extremities a, b of the interval, we can easily prove the existence of the quantity L_1 described in (β) by breaking up the interval ab into sub-intervals ac_1, c_1c_2, $\cdots$, c_kb. Then it is clear from the special case we have just treated that we can find constants L', L'', L''', $\cdots$, $L^{[k+1]}$ such that:

(1) when $L' \leqq \lambda < L$ the solution of (2) for which $y(a, \lambda) = \alpha$ and $y'(a, \lambda) = \alpha'$ has no root in the interval $a < x \leqq c_1$ and satisfies the relation $y'(c_1, \lambda) / y(c_1, \lambda) > 1$;

(2) when $L^{[i]} \leqq \lambda < L$ ($i = 2, 3, \cdots, k$) the solution of (2) for which $y'(c_{i-1}, \lambda) / y(c_{i-1}, \lambda) = 1$ has no root in the interval $c_{i-1} \leqq x \leqq c_i$ and satisfies the relation $y'(c_i, \lambda) / y(c_i, \lambda) > 1$;

(3) when $L^{[k+1]} \leqq \lambda < L$ the solution of (2) for which $y'(c_k, \lambda) / y(c_k, \lambda) = 1$ has no root in the interval $c_k \leqq x \leqq b$ and (when $\beta \neq 0$) satisfies the relation

$$y'(b, \lambda) / y(b, \lambda) > \beta' / \beta.$$

If we let L_1 be the largest of the quantities L', L'', $\cdots$, $L^{[k+1]}$ the truth of (β) follows at once from Theorems I and II of the present paper.

§2. *Klein's Generalized Theorem of Oscillation.*

We start here from the differential equation (1) in which we will suppose the coefficient q to have the form:

$$q(x) = \chi(x) + \psi(x) [C_k x^k + C_{k-1} x^{k-1} + \cdots + C_1 x + C_0],$$

and we will suppose that the functions $p(x)$, $\chi(x)$, and $\psi(x)$

do not involve the parameters $C_0, C_1, \cdots, C_k$. Let there be given $k+1$ finite intervals $a_0b_0, a_1b_1, \cdots, a_kb_k$ such that:

$$a_0 < b_0 \leqq a_1 < b_1 \leqq a_2 < b_2 \cdots < b_{k-1} \leqq a_k < b_k,$$

and let us assume that throughout each of the intervals $a_i \leqq x \leqq b_i$ $(i = 0, 1, \cdots, k)$ the functions $p(x)$, $\chi(x)$, and $\psi(x)$ are single valued, continuous, and real, and that $\psi(x)$ does not change sign in any one of these intervals nor have an infinite number of roots in them. Then we may state Klein's generalized theorem of oscillation as follows:

IV. *There is one and only one real determination of the parameters $C_0, C_1, \cdots, C_k$ for which the equation (1) has $k+1$ solutions $y_0, y_1, \cdots, y_k$ such that $y_i (i = 0, 1, \cdots, k)$ has just m_i roots in the interval $a_i < x < b_i$ and satisfies the conditions:*

$$\begin{aligned} a_i' y_i(a_i) - a y_i'(a_i) &= 0, \\ \beta_i' y_i(b_i) - \beta y_i'(b_i) &= 0. \end{aligned} \tag{3}$$

In this theorem the quantities m_i are any integers positive or zero and $a_i, a_i', \beta_i, \beta_i'$ are any real quantities subject only to the restriction that a_i and a_i' are not both zero and that β_i and β_i' are not both zero.

In order to prove the above theorem we will introduce the variables:

$$t_i = g_i \int_{\xi_i}^{x} e^{-\int_{\xi_i}^{x} p\,dx}\,dx + f_i \quad (i = 0, 1, \cdots, k)$$

where g_i and f_i are real constants which may be arbitrarily chosen provided $g_i \neq 0$, and ξ_i is any real constant subject to the condition $a_i \leqq \xi_i \leqq b_i$. It should be noticed that the variable t_i as thus defined is real when $a_i \leqq x \leqq b_i$, and that this interval of the x-axis corresponds in a one to one manner to a finite interval $\sigma_i\tau_i$ of the t_i-axis. Written in terms of t_i, (1) has the form:

$$\frac{d^2y}{dt_i^2} = \varphi_i(x) y, \tag{4}$$

where:

$$\varphi_i(x) = \frac{-1}{g_i^2}\,\chi(x) e^{2\int_{\xi_i}^{x} p\,dx} - \frac{1}{g_i^2}\,\psi(x) e^{2\int_{\xi_i}^{x} p\,dx}\,[C_k x^k + \cdots + C_0].$$

It is important to bear in mind in the following work that if a solution y_i of (1) regarded as a function of x has m_i roots in the interval $a_i < x < b_i$ and satisfies the two con-

ditions (3), the same function regarded as a function of t_i will be a solution of (4) having m_i roots in the interval $\sigma_i < t_i < \tau_i$ (or $\tau_i < t_i < \sigma_i$) and will satisfy relations of the form:

$$\begin{aligned} &\alpha_i' y_i(\sigma_i) - \bar{\alpha}_i y_i'(\sigma_i) = 0, \\ &\beta_i' y_i(\tau_i) - \bar{\beta}_i y_i'(\tau_i) = 0, \end{aligned} \tag{5}$$

where $y' = \frac{dy}{dt_i}$ and where $\bar{\alpha}_i$ and $\bar{\beta}_i$ are constant (not zero) multiples of α_i and β_i; and that the converse of the statement just made is also true.

It is easily seen that if $k = 0$ Theorem IV is merely a special case of Theorem III since then if ψ is negative in the interval $a_0 b_0$ $\varphi_0(x)$ will continually increase (or rather never decrease) for every point of this interval as C_0 increases from $-\infty$ to $+\infty$ and will, except for the points where $\psi(x) = 0$, increase from $-\infty$ to $+\infty$; while if ψ is positive the same will be true as $-C_0$ increases from $-\infty$ to $+\infty$. Theorem IV is therefore true if there is only one parameter and one interval.

Let us then, following the ordinary method of mathematical induction, assume that Theorem IV is true for k parameters and k intervals. If from this assumption we can deduce the theorem for $k+1$ parameters and intervals the proof of the theorem will be complete.

Since in (1) we may write:

$$q(x) = \bar{\chi}(x) + \psi(x)\,[C_{k-1}x^{k-1} + \cdots + C_0],$$

where

$$\bar{\chi}(x) = \chi(x) + C_k x^k \psi(x),$$

it follows (from the assumption we have made that our theorem holds when there are k parameters) that, C_k having an arbitrarily assigned fixed value, there is always one and only one real set of values $C_0, C_1, \cdots, C_{k-1}$ for which (1) has k solutions $y_0, y_1, \cdots, y_{k-1}$ satisfying in the intervals $a_i b_i (i = 0, 1, \cdots, k-1)$ the desired conditions. From this point on we will assume that $C_0, C_1, \cdots, C_{k-1}$, have thus been determined as single valued real functions of C_k. It remains then merely to show that there is one and only one value of C_k for which (1) has a solution y_k which satisfies the desired conditions in the interval $a_k b_k$.

In order to indicate the dependence of φ_i on C_k we will write $\varphi_i(x, C_k)$. Let us now assign to C_k two different values C_k' and C_k'' and consider the difference:

$$\varphi_i(x, C_k') - \varphi_i(x, C_k'')$$

$$= \frac{-1}{g_i^2}\psi(x) e^{2\int_{\xi_i}^{x}\rho dx}[(C_k' - C_k'')x^k + \cdots + (C_0' - C_0'')].$$

Since by hypothesis each of the equations:

$$\frac{d^2y}{dt_i^2} = \varphi_i(x, C_k')y \text{ and } \frac{d^2y}{dt_i^2} = \varphi_i(x, C_k'')y$$

has a solution with m_i roots in the interval $\sigma_i < t_i < \tau_i$ $(i = 0, 1, \cdots, k-1)$ and since these two solutions satisfy the same conditions (5) it follows at once from the theorems of comparison (*Th. of Osc. 1*, Theorems I and II) that the above difference must change sign in the interval $\sigma_i < t_i < \tau_i$ that is in the interval $a_i < x < b_i$. It follows that the polynomial:

$$(C_k' - C_k'')x^k + (C_{k-1}' - C_{k-1}'')x^{k-1} + \cdots + (C_0' - C_0'')$$

must have a root x_i in this interval. Accordingly when $i = 0, 1, \cdots, k$:

$$\varphi_i(x, C_k') - \varphi_i(x, C_k'')$$

$$= \frac{-1}{g_i^2}\psi(x) e^{2\int_{\xi_i}^{x}\rho dx}(C_k' - C_k'')(x - x_0)(x - x_1)\cdots(x - x_{k-1}), \tag{6}$$

where $a_j < x_j < b_j$ $(j = 0, 1, \cdots, k-1)$. From this fundamental formula we can deduce several important results:

(*a*) *$\varphi_k(x, C_k)$ is when $a_k \leqq x \leqq b_k$ and for all values of C_k a continuous function of (x, C_k) and therefore of (t_k, C_k).*

For in the first place it is clear that it is a continuous function of x, so that however small the positive quantity ε may be, we can choose the positive quantity δ_1 so small that when $|x - \overline{x}| \leq \delta_1$, $|\varphi_k(\overline{x}, C_k) - \varphi_k(x, C_k)| < \varepsilon/2$. Moreover from (6) $|\varphi_k(\overline{x}, C_k) - \varphi_k(\overline{x}, \overline{C}_k)| < K|C_k - \overline{C}_k|$ where K is a positive constant, independent of $\overline{x}$, C_k and $\overline{C}_k$.* From the last formula it is clear that however small ε may be, we can choose a positive quantity δ_2 so small that when

$$|C_k - \overline{C}_k| < \delta_2, \qquad |\varphi_k(\overline{x}, C_k) - \varphi_k(\overline{x}, \overline{C}_k)| < \varepsilon/2.$$

* We need merely to take K greater than the greatest value of:

$$\frac{1}{g_k^2}|\psi(x)|e^{2\int_{\xi_k}^{x}\rho dx}(x - \sigma_0)(x - \sigma_1)\ldots(x - \sigma_{k-1})$$

in the interval $a_k \leqq x \leqq b_k$.

Accordingly letting δ denote the smaller of the two quantities δ_1 and δ_2 we see that when

$|x - \bar{x}| < \delta$ and $|C_k - \bar{C}_k| < \delta$, $|\varphi_k(x, C_k) - \varphi_k(\bar{x}, \bar{C}_k)| < \varepsilon$

that is $\varphi_k(x, C_k)$ is a continuous function of (x, C_k) as was to be proved.

(β) *If* $C_k' > C_k''$ *then* $\varphi_k(x, C_k') \geqq \varphi_k(x, C_k'')$* $(a \leqq x \leqq b_k)$ *provided that* $\psi(x)$ *is negative in this interval; otherwise* $\varphi_k(x, C_k') \leqq \varphi_k(x, C_k'')$.*

(γ) *Except at the points at which* $\psi(x) = 0$ *and perhaps the point* $x = a_k$ *we have at every point* x *of the interval* $a_k \leqq x \leqq b_k$:

$$\lim_{C_k = \infty} \varphi_k(x, C_k) = \infty .$$

The truth of (β) and (γ) follows immediately from (6).

Accordingly $\varphi_k(x, C_k)$ is at every point of the interval $a_k \leqq x \leqq b_k$, *i. e.*, of $\sigma_k \leqq t_k \leqq \tau_k$, a continuous function of (t_k, C_k) which, except at a finite number of points, continually increases with C_k(or $-C_k$), and except at a finite number of points increases from $-\infty$ to $+\infty$ as C_k(or $-C_k$) varies from $-\infty$ to $+\infty$. Sturm's generalized theorem of oscillation (Theorem III) tells us therefore that there is one and only one value of C_k for which (4) has a solution satisfying the desired conditions in the interval $\sigma_k \tau_k$ and the proof of Theorem IV is complete.

We proceed now to a generalization of the preceding theorem:

V. *In Theorem IV we may at some or at all of the points* a_i, b_i *drop the requirement that the functions* $p(x)$, $\chi(x)$, *and* $\psi(x)$ *be continuous provided that as* x *approaches these points the functions* t_i *and* φ_i *approach finite limits.*† *We must, however, then at these points replace conditions* (3) *by conditions* (5), *the two sets of conditions being then no longer equivalent.*

It is not necessary to give any new proof of this theorem as the proof just given of Theorem IV applies at once since what is essential in that proof referred not to the form (1) of the differential equation but to the binomial form (4).

We proceed now to recall certain important special cases of (1) to which Theorems IV and V apply:

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* The equality sign holding at most for a finite number of values of x namely for the points if any at which $\psi(x) = 0$.

† I should like to add here that we need not even require that the limit of ϕ_i be finite, all that is necessary being that ϕ_i do not become infinite too rapidly as x approaches a_i and b_i. The exact statement of the conditions here as well as all proof I must, however, reserve for a future occasion.

The most general homogeneous linear differential equation of the second order with rational coefficients which is everywhere regular and has as its finite singular points the points e_i $(i = 1, 2, \cdots, n)$ with exponents $\varkappa_i'$, $\varkappa_i''$ is:*

$$(7)\quad \frac{d^2y}{dx^2} + \left(\frac{1-\varkappa_1'-\varkappa_1''}{x-e_1} + \cdots + \frac{1-\varkappa_n'-\varkappa_n''}{x-e_n}\right)\frac{dy}{dx}$$
$$+ \frac{1}{f(x)}\left[\frac{\varkappa_1'\varkappa_1''f'(e_1)}{x-e_1} + \cdots + \frac{\varkappa_n'\varkappa_n''f'(e_n)}{x-e_n}\right.$$
$$\left. + C_{n-2}x^{n-2} + C_{n-3}x^{n-3} + \cdots + C_0\right] y = 0,$$

where $$f(x) = (x-e_1)(x-e_2)\cdots(x-e_n)$$

and $f'(x) = df(x)/dx$. The point $x = \infty$, the product of whose exponents is O_{n-2}, may be singular or non-singular.

We will suppose that the coefficients of this equation are real for real values of x. For this to be the case it is necessary and sufficient that:

(a) the singular points $e_1, e_2, \cdots, e_n$ be conjugate imaginaries in pairs in so far as they are not real;

(b) the exponents $\varkappa_i'$, $\varkappa_i''$ of a real singular point e_i be either real or conjugate imaginaries;

(c) the exponents $\varkappa_i'$, $\varkappa_i''$ and $\varkappa_j'$, $\varkappa_j''$ of two conjugate imaginary singular points e_i and e_j be so related that $\varkappa_i'$ is the conjugate of one and that $\varkappa_i''$ is the conjugate of the other of the two quantities $\varkappa_j'$, $\varkappa_j''$;

(d) the quantities $C_{n-2}, C_{n-3}, \cdots, C_0$ be real.

Let us now take $k+1$ intervals $a_i b_i$ $(i = 0, 1, \cdots, k)$ on the real x-axis as on p. 371, *i. e.*, so that no two of these intervals overlap and none of the points $e_1, e_2, \cdots, e_n$ lie within or at an extremity of any of these intervals. If we suppose that $k \leqq n-2$ we see that *we can determine the parameters* $C_2, C_{k-1}, \cdots, C_0$ *by means of the Theorem of Oscillation IV.*

Let us now assume that in equation (7) one of the singular points, which for the sake of simplicity we will take as e_1, is real and that $\varkappa_1'' = 0$ while $\varkappa_1'$ is real and satisfies the inequality $0 < \varkappa_1' \leqq \frac{1}{2}$. Then if one of the intervals $a_i b_i$ has an extremity at e_1, we see that t_i and φ_i approach finite limits as x approaches e_1, so that we see by Theorem V that *we can still determine the parameters* $C_2, C_{k-1}, \cdots, C_0$ *by means of the theorem of oscillation.*

*Cf. for instance my book: "Ueber die Reihenentwickelungen der Potentialtheorie," pp. 112-113.

It goes without saying that any number of the extremities of the intervals $a_i b_i$ $(i = 0, 1, \cdots, k)$ may lie at singular points provided that the conditions above stated for the point e_1 are satisfied at each of these points.

It should be noticed that the generalized Lamé's equation, whether looked at from Heine's* or from Klein's† standpoint, has as the exponents of each of its finite singular points the values $0, \frac{1}{2}$, so that in this case any or all of the intervals $a_i b_i$ may reach up to singular points.‡

We will note in conclusion that the cases we have just mentioned by no means exhaust the important applications of Theorems IV and V. For instance, the degenerate forms of Lamé's equation where two or more singular points coincide come immediately under Theorem IV.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

THE CONSTRUCTION OF SPECIAL REGULAR RETICULATIONS ON A CLOSED SURFACE.

BY PROFESSOR HENRY S. WHITE.

(Read in part at the Mathematical Conference in Chicago, January 1, 1897, and in final form at the Meeting of the American Mathematical Society, April 30, 1898.)

INTRODUCTORY.

THE reticulations whose existence is here to be discussed are called regular because of two properties: the number of termini of edges assembled in one vertex is the same for all vertices of the reticulation, and the number of edges in the boundary of a face is the same for all faces. These two numbers, r and s, together with p, the deficiency of the supporting surface, shall be assumed to characterize the reticulation sufficiently for present purposes. Of regular reticulations classified on this basis, only a finite number of classes are possible on a surface of given deficiency. Some of these possible classes, if $p > 2$, are derivable from those of lower deficiency; those not so derivable are properly

* Handbuch der Kugelfunctionen, vol. I., p. 445.

† Cf. my book: "Ueber die Reihenentwickelungen der Potentialtheorie," p. 114.

‡ It is in fact easy to see that they may turn back at the singular points and thus cover parts of the x-axis more than once. Cf. p. 123 of my book just referred to.

called *special* or primitive. In a paper* read at the Buffalo meeting of the AMERICAN MATHEMATICAL SOCIETY I explained this distinction, and showed tables of all primitive regular reticulations for deficiencies 2 and 3. Those tables had only a restrictive, not a positive significance; and fortunately I was able to show also sets of models proving empirically the existence of types of all classes admitted by the tables.

Further study shows that there are certain connections among the tabulated numbers, rendering probable the derivation of types of all the classes belonging to the same surface from a single fundamental reticulation, *provided that the deficiency of the surface is lower than 9.* Professor Ellery W. Davis has demonstrated the occurrence of exceptional pairs of characteristics r, s, devoid of such connection, and has thus shown critically one of the limitations of the present discussion. Excluding such *Davis reticulations*, we find our problem reduced to that of the construction of one fundamental reticulation for each surface. This can always be effected, by a method due to Riemann, the so-called canonical dissection of a Riemann surface, with certain obviously admissible modifications. The method allows, moreover, an examination of the grouping of the several faces about each vertex; or dually, of the vertices about each face. While it is not proposed here to develop these minutiæ, it is desired to give a working outline of three essential parts of the systematic verification of each table of admissible reticulations:

1. The mutual derivations of two dual reticulations from each other;
2. Two (dual) processes for multiplying the number of vertices or faces;
3. The dissection of a Riemann surface into a fundamental polygon.

To obviate confusion it should be understood that the regularity of these reticulations is not the same as that defined by Dyck and Klein in function-theoretic investigations; the two definitions overlap, but neither includes the other.

§ 1. DUALITY.

Two reticulations are dual to each other when the numbers r, s, V, F, of the first are equal respectively to s', r', F', V', of the second; and when each face of the one corre-

*Printed in BULLETIN for December, 1896, pp. 116–121.

sponds to a vertex of the other in such a way that the succession of vertices about each face corresponds exactly to the succession of faces about the corresponding vertex. From this definition it is clear that if one of two dual reticulations is regular, the other must be so. Either may be derived from the other, so far as its numerical or tactical character is concerned, in the following way: In each face of the first, assume a point for a vertex of the second. On every edge of the first, mark a point where an edge of the second shall cross it. Connect each new vertex to the designated points on all edges of the face in which it lies, and consider as one continuous line every pair of these join-lines which meet at an edge of the first reticulation. Then the new vertices and lines will constitute a second reticulation dual to the first. For the numbers r and s of the first are by construction equal to s' and r' of the second, likewise V and F to F' and V' respectively. And since two faces in contact along an edge are replaced by two vertices terminating a new edge, and *vice versa*, therefore the order of succession about each face and about each vertex has been strictly preserved, as the definition requires.

The same freedom to substitute one of two dual figures for the other, which is allowed in projective geometry, is therefore allowed equally in every purely topological study of reticulations. Of practical importance is this observation only when, of two dual reticulations whose existence is to be proved, the one is somewhat the easier for the intuition to grasp.

§2. Multiplying the Number of Faces or of Vertices.

If a regular reticulation is given, and the tables show as possible a second which has the same number of vertices, while the number of faces is an exact multiple of that in the first, the obvious way of deriving the second from the first is by drawing diagonals from vertex to vertex as often as necessary. This does not change the order of sequence among faces already meeting at any vertex, but inserts one new face with every diagonal drawn from that vertex. In each face, the initial point of each required diagonal may be arbitrary, and hence a comparatively large number of alternatives may have to be tried before finding one that satisfies the obvious restriction: *that all vertices are to become termini of the same number of diagonals.* A finite number of trials will always prove whether it is possible to effect the proposed modification upon a reticulation of given type.

Reversing now the significance of the numbers, suppose

that a second reticulation is to be derived from the first by retaining the number of faces but multiplying the number of its vertices. For this purpose the edges may be thought of as elastic fibres, the termini as loops or eyelets sliding freely upon all other edges. At each vertex of the given reticulation the termini are to be divided into groups of suitable size—the extremes larger by one than the intermediate groups; and each group is thought of as sliding away from its next preceding neighboring group along some edge apportioned to it. The point where it rests becomes a new terminus. Otherwise the termini at a vertex may be regarded as clustered along a small but extensible circle, which is to be cut at some selected point and stretched until the segments of arc between clusters become edges of finite length. Each such intercalated edge augments by one the number of boundaries of two faces; and the restriction is like that imposed a moment ago upon the dual operation, *that all faces must have the number of their edges augmented alike.*

TABLES OF SPECIAL REGULAR RETICULATION-NUMBERS.

Deficiency :	4,		5,		6,		7,		8,		9.	
	1_{16}	1_{16}	1_{20}	1_{20}	1_{24}	1_{24}	1_{28}	1_{28}	1_{32}	1_{32}	1_{36}	1_{36}
	1_{18}	2_9	1_{22}	2_{11}	1_{26}	2_{13}	1_{30}	2_{15}	1_{34}	2_{17}	1_{38}	2_{19}
	1_{28}	7_4	1_{24}	3_8	1_{44}	11_4	1_{52}	13_4	1_{36}	3_{12}	1_{68}	17_4
	2_{10}	2_{10}	1_{30}	6_5	1_{66}	22_3	1_{78}	26_3	1_{40}	5_8	1_{102}	34_3
	2_{12}	4_6	1_{36}	9_4	2_{14}	2_{14}	2_{21}	7_6	1_{42}	6_7	2_{21}	3_{14}
	2_{16}	8_4	1_{54}	18_3	2_{15}	3_{10}			1_{50}	10_5	2_{27}	9_6
	2_{24}	16_3	2_{15}	5_6	2_{16}	4_8			1_{60}	15_4	3_{38}	38_3
	4_7	4_7	3_{22}	22_3	2_{18}	6_6			1_{90}	30_3	5_{14}	14_5
	4_{10}	10_4	6_7	7_6	2_{20}	8_5			2_{18}	2_{18}	6_{11}	11_6
	4_{15}	20_3			2_{24}	12_4			2_{20}	4_{10}		
	8_7	14_4			2_{26}	24_3			2_{24}	8_6		
					3_{26}	26_3			2_{22}	16_4		
					4_9	4_9			2_{48}	32_3		
					4_{14}	14_4			3_{34}	34_3		
					4_{21}	28_3			4_{11}	4_{11}		
					6_8	8_6			4_{12}	6_8		
					6_{16}	32_3			4_{15}	12_5		
					8_9	18_4			4_{18}	18_4		
					12_{11}	44_3			4_{27}	36_3		
									6_{10}	10_6		
									6_{20}	40_3		
									8_{11}	22_4		
									10_8	16_5		
									12_{12}	52_3		

These processes may suffice for verifying empirically any given restrictive table if in the first place a fundamental polygon on the surface is given, and if in the second place the table shows each pair of values, V and F, to have one value in common with a smaller pair. This latter condition is fulfilled in the tables previously published for $p = 2$ and $p = 3$, also in the summary on the preceding page for $p = 4, 5, 6, 7, 8$, *but not for* $p = 9$. Each set of numbers belonging to one reticulation is given in the order F, V. The last two in column 9 are specials of the *Davis* type.

§ 3. Dissection of a Riemann Surface Yields a Fundamental Polygon.

Any finite closed surface may be regarded as an ideal Riemann surface. Disregarding its historical relation to the theory of functions, we wish to avail ourselves of a well-known method belonging strictly to topology or *analysis situs*, for laying out circuits or lines of section which shall reduce a multiply-connected surface to a simply-connected one, with a continuous boundary. If the deficiency of the surface is p, that means that exactly p circuits can be drawn in it which shall be non-collapsible (irreducible) and non-intersecting, and which shall not divide the surface into separate portions. After that, p conjugate circuits can be drawn, each of which shall be non-collapsible, and shall meet one, and only one, of the former circuits, and that in but a single point; these circuits also leave the surface still one connected whole. If now lines be drawn, connecting in series all p points where pairs of conjugate circuits intersect, the dissection is complete, and the resulting surface is simply-connected with a single continuous boundary, consisting of all the circuits and join-lines, each traversed twice in opposite senses.

For our use, a better statement of the theorem is this: *Through any point in the surface* $2p$ *non-collapsible circuits can be drawn, each of which shall be cut in that point by at least one other, and which shall not intersect elsewhere. Such* $2p$ *circuits cut the surface into one simply-connected polygon.*

This is evidently a fundamental polygon such as we require. It has 1 vertex and $2p$ edges. In the vertex are $4p$ termini, and its boundary has $4p$ consecutive edges. Comparing this with the formulæ

$$V = \frac{(4p-4)\cdot s}{(r-2)(s-2)-4}, \quad F = \frac{(4p-4)\cdot r}{(r-2)(s-2)-4},$$

we see that $r = s = 4p$ will give the values $V = 1$, $F = 1$, as expected. We conclude: *There is therefore on every closed (two-sided) surface of deficiency p a fundamental polygon of one vertex and having 4p edges united in pairs.*

It is now an easy matter, by either of the two processes described above, to cut this polygon by diagonals or intercalated edges, into as many faces with one vertex, or vertices upon one face, as there are factors in $4p - 2$. For if we set in the formulæ cited above:

$$4p - 2 = g(s - 2),$$
$$4p + 2g - 4 = r - 2,$$

we shall find $V = 1$, $F = g =$ any factor of $4p - 2$.
This establishes the existence of all tabulated reticulations having either V or $F = 1$. For others, apparently derivable from these, the verification needs in each case only a finite number of trials. There would arise further theoretical questions, of whose solution experiment may be expected to furnish some suggestion. Before leaving this topic, it is worth while to add a remark, which in many trials leads to the highest attainable degree of symmetry—a remark upon the arrangement of edges about the vertex of the fundamental polygon.

The order of the edges about the vertex of the fundamental polygon, or any derivative of it, is arbitrary, under the sole restriction that between the opposite termini of each several edge there shall occur a terminus of at least one edge, whose other terminus does not occur in the same interval, and all termini must be divisible thus, in at least one way, into mutually exclusive sets of four. In other words, in the succession of termini about the afore-mentioned vertex, each pair of termini belonging to the same edge must be separated by the two termini of at least one other edge. This restriction, it will be seen, is always satisfied by arranging in succession one terminus from each of the edges, followed by the opposite termini in the same order. Thus each circuit is cut by *all* the others; this renders also the fundamental polygon dual in type to itself.

After so much of description of the processes employed, it is easy for any one to formulate a notation which shall exhibit at each stage of the derivation process:

1. A definite mark for each edge, this mark to remain the same through all stages after its introduction;

2. A mark for each vertex, and a list of the edges and faces in their order of grouping about each vertex;

3. A mark for each face, and a list of the edges and vertices in their order upon the boundary of each face.

Such a notation must contain a mark of distinction for the two sides of an edge; an easy matter if the direction of positive rotation be adopted uniformly in listing arrangements about the vertices and faces respectively.

These processes, and the proved existence of fundamental polygons, open a range of particular problems of considerable interest. But of even superior interest must be, at least until it is solved, *the problem of finding a method for constructing, a priori, upon a given surface the exceptional (Davis) special reticulations whose characteristics are given by the restrictive tables.*

NORTHWESTERN UNIVERSITY,
April, 1898.

SYSTEMS OF SIMPLE GROUPS DERIVED FROM THE ORTHOGONAL GROUP.

BY DR. L. E. DICKSON.

1. IN the February number of the BULLETIN I determined the order ω of the group G of orthogonal substitutions of determinant unity on m indices in the $GF[p^n]$ and proved that, for* $p^n > 5$, $p \neq 2$, the group is generated by the substitutions

$$O^{\alpha,\beta}_{ij}: \begin{cases} \xi_i' = \alpha\xi_i + \beta\xi_j \\ \xi_j' = -\beta\xi_i + \alpha\xi_j \end{cases} \quad (\alpha^2 + \beta^2 = 1).$$

The structure of G was determined for the case $p = 2$. I have since proved† that for every $m > 4$ and every $p^n > 5$ of the form $8l + 3$ or $8l + 5$, the factors of composition of G

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* The fact that $p^n = 3$ is an exception was not pointed out in the BULLETIN. In fact Jordan had not proven case 2° of § 211 when $-1 =$ square, so that the case $a^2 = b^2 = c^2 = \ldots = 1$ was unsolved when $p = 3$, $m = 3k + 1$. The theorem is readily proven when $p^n = 3^n$, $n > 1$; but for $p^n = 3$ an additional generator is necessary and sufficient, viz.,

$$W = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \quad W^3 = 1.$$

† A preliminary account was presented before the Mathematical Conference at Chicago, December 30, 1897.

are 2 and $\omega/2$ for m odd, and 2, 2, $\omega/4$ for m even. [For later results on the cases thus excluded see §§ 11–15.]

For m odd, the orthogonal group G had the same order and factors of composition as the linear Abelian group on $m-1$ indices. Judging from the results for the corresponding continuous groups of Lie, the resulting triply-infinite systems of simple groups of the same order $\omega/2$ are probably isomorphic when $m=3$, but not when $m>3$. [See § 14.]

Excluding here the cases $m=5$, 6, 7, which require lengthy special investigations, I will now give a short, simple proof of the above result. The complete memoir will appear in the *Proceedings of the California Academy of Sciences.**

The substitutions $O_{1,2}^{\alpha,\beta}$ form a commutative group of order† $p^n \pm 1$. A subgroup of index 2 is formed by the substitutions

$$Q_{1,2}^{\alpha,\beta}: \quad \begin{aligned} \xi_1' &= (\alpha^2-\beta^2)\,\xi_1 - 2\alpha\beta\xi_2, \\ \xi_2' &= 2\alpha\beta\xi_1 + (\alpha^2-\beta^2)\xi_2. \end{aligned}$$

Indeed,

$$Q_{1,2}^{\gamma,\delta}\, Q_{1,2}^{\alpha,\beta} = Q_{1,2}^{\alpha\gamma-\beta\delta,\ \alpha\delta+\beta\gamma}.$$

Further, its order is $\frac{1}{2}(p^n \pm 1)$ since $Q_{1,2}^{\alpha,\beta}$ and $Q_{1,2}^{\gamma,\delta}$ are identical if and only if $\alpha = \pm\gamma$, $\beta = \pm\delta$.

If C_i denotes the substitution affecting only the index ξ_i, whose sign it changes, $C_1 C_2$ is always contained among the substitutions $Q_{1,2}^{\alpha,\beta}$.

Since we suppose that $p^n = 8l \pm 3$, 2 is a not-square. Thus, with $\alpha^2+\beta^2=1$, we cannot have $\alpha^2-\beta^2=0$. Hence if T_{12} denotes the transposition $(\xi_1\xi_2)$, $T_{12}C_1$ is not of the form $Q_{1,2}^{\alpha,\beta}$ but serves to extend the group of the latter to the group of the $O_{1,2}^{\alpha,\beta}$. Furthermore, if $j>2$, $T_{12}C_1$ transforms $Q_{1j}^{\alpha,\beta}$ into $Q_{2,j}^{\alpha,\beta}$ and $Q_{2j}^{\alpha,\beta}$ into $Q_{1,j}^{\alpha,-\beta}$. Hence if we extend the alternating group on the m letters ξ_i by the substitutions $Q_{1,2}^{\alpha,\beta}$ we obtain a group H of index 2 under G.

3. THEOREM: *For $p^n>5$, $m>7$, the maximal invariant subgroup of H is of order 2 or 1 according as m is even or odd.*

For m even, H contains an invariant subgroup of order 2 generated by the substitution

$$N: \qquad \xi_i' = -\xi_i \quad (i=1,\cdots,m).$$

Suppose H has an invariant subgroup I containing a substitution

* Third Series, vol. 1, No. 4; the later results in No. 5.

† "Orthogonal Group in a Galois Field," § 3; BULLETIN, February, 1898, pp. 196–200.

$$S: \quad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \quad (i=1,\cdots,m),$$

neither the identity nor N. By suitably transforming S, we can suppose that $a_{11}^2+a_{21}^2+1$. Then S is not commutative with C_1C_2; for, if so, it would be merely a product of a substitution affecting only ξ_1 and ξ_2 by a substitution affecting only $\xi_3, \cdots, \xi_m$. Hence the group I contains the substitution, not the identity,

$$S^{-1}\,C_1\,C_2\,S\,C_1\,C_2 = S_\alpha C_1\,C_2,$$

where $S_\alpha \equiv S^{-1}\,C_1\,C_2\,S$, of period two, is found to be

$$\xi_i' = \xi_i - 2a_{i1}\sum_{j=1}^{m} a_{j1}\xi_j - 2a_{i2}\sum_{j=1}^{m} a_{j2}\xi_j \quad (i=1,\cdots,m).$$

4. *Lemma:* The orthogonal substitution

$$O_{ijk}: \quad \begin{aligned} \xi_i' &= \lambda\ \xi_i + \mu\ \xi_j + \nu\ \xi_k, \\ \xi_j' &= \lambda'\,\xi_i + \mu'\,\xi_j + \nu'\,\xi_k, \\ \xi_k' &= \lambda''\xi_i + \mu''\xi_j + \nu''\xi_k, \end{aligned}$$

where $\lambda^2+\mu^2+\nu^2=1$, $\lambda\lambda'+\mu\mu'+\nu\nu'=0$, etc., transforms S_α into $S_{\alpha'}$ where

$$\begin{aligned} a_{il}' &= \lambda\ a_{il} + \mu\ a_{jl} + \nu\ a_{kl}, \\ a_{jl}' &= \lambda'\,a_{il} + \mu'\,a_{jl} + \nu'\,a_{kl}, \qquad (l=1,2) \\ a_{kl}' &= \lambda''a_{il} + \mu''a_{jl} + \nu''a_{kl}, \\ a_{sl}' &= a_{sl}, \qquad (s=1,\cdots,m,\ s+i,j,k). \end{aligned}$$

If $a_{j1}+0$, we can choose λ, μ, ν such that $a_{j1}'=0$. For if $a_{i1}^2+a_{j1}^2=0$ and therefore $a_{j1}+0$, we may take

$$\lambda = \frac{-\,a_{k1}}{2a_{i1}}, \quad \mu = \frac{-\,a_{k1}}{2a_{j1}}, \quad \nu = 1.$$

If $a_{i1}^2+a_{j1}^2+0$, we derive the equivalent condition,

$$\{\mu(a_{i1}^2+a_{j1}^2)+\nu a_{j1}\,a_{k1}\}^2+\nu^2a_{i1}^2(a_{i1}^2+a_{j1}^2+a_{k1}^2)=a_{i1}^2(a_{i1}^2+a_{j1}^2),$$

which has solutions* for μ and ν in the $GF[p^n]$ except when $a_{i1}^2+a_{j1}^2+a_{k1}^2=0$. In the latter case the condition $\lambda^2+\mu^2+\nu^2=1$ may be written

$$a_{i1}^2 = -\,(\mu a_{k1}-\nu a_{j1})^2,$$

having solutions if and only if -1 be a square.

* BULLETIN, l.c. § 3.

5. *Lemma:* O_{ijkl} transforms S_α into $S_{\alpha'}$, where

$$a_{i1}' = \lambda a_{i1} + \mu a_{j1} + \nu a_{k1} + \sigma a_{l1}.$$

If $a_{i1}^2 + a_{j1}^2 + a_{k1}^2 = 0$, the values

$$\lambda = \frac{-a_{i1}}{a_{i1}}, \quad \mu = \frac{a_{k1}a_{l1}}{a_{i1}^2}, \quad \nu = \frac{-a_{l1}a_{j1}}{a_{i1}^2}, \quad \sigma = 1$$

make $\qquad a_{i1}' = 0, \quad \lambda^2 + \mu^2 + \nu^2 + \sigma^2 = 1$

6. The invariant subgroup I of H was shown to contain the substitution $S' \equiv S_\alpha C_1 C_2$ not the identity. Transforming S' successively by

$$O_{i345} \text{ or } T_{12} C_1 O_{i345} \qquad (i = m, m-1, \cdots, 6),$$

according as the one or the other belongs to H, we obtain, by §§ 4–5, a substitution $S_{\alpha'} C_1 C_2$ belonging to I and having $a_{m1}' = a_{m-1\,1}' = \cdots = a_{61}' = 0$. Also, by § 4, we may make $a_{51}' = 0$; for we have

$$a_{11}'^2 + a_{21}'^2 = a_{11}^2 + a_{21}^2 + 1,$$

so that $\qquad a_{51}'^2 + a_{41}'^2 + a_{31}'^2 + 0.$

Next we transform $S_\alpha \cdot C_1 C_2$ successively by

$$O_{j567} \quad (j = m, m-1, \cdots, 8)$$

and obtain in I a substitution $S_{\alpha''} C_1 C_2$ having

$$a_{j1}'' \equiv a_{j1}' = 0, \quad a_{j2}'' = 0 \quad (j = m, \cdots, 8).$$

The group I thus contains a substitution

$$S: \qquad \xi_i' = \sum_{j=1}^{7} \beta_{ij} \xi_j \quad (i = 1, \cdots, 7)$$

neither the identity nor $N \equiv C_1 C_2 \cdots C_m$.

7. If S be commutative with every C_i, it is merely a product of an even number of the C_i, in which certain ones as C_k are lacking. But if

$$S = C_i C_j C_r C_s C_t \cdots,$$

the group I contains

$$T_{ij} T_{rs} S T_{rs} T_{ij} S^{-1} = C_k C_j,$$

and hence, by transforming by suitable even substitutions, every product of two C's. But H contains either $O_{1,2}^{\alpha,\beta}$ or $O_{1,2}^{\alpha,\beta} T_{12} C_1$, which transform $C_1 C_3$ into $Q_{1,2}^{\alpha,\beta} C_1 C_3$ and $Q_{1,2}^{\alpha,\beta} C_2 C_3$

respectively. Hence the group I contains every $Q^{\alpha,\beta}_{1,2}$, among which, if $p^n > 5$, occurs one different from the identity and from C_1C_2.

8. We may thus assume that S is not commutative with C_1, for example. Supposing $m \geqq 8$, S is commutative with C_8. Hence the group I contains the substitution not the identity

$$S^{-1}C_1C_8SC_1C_8 = R_\beta C_1,$$

where $R_\beta \equiv S^{-1}C_1S$ is seen to be

$$R_\beta: \quad \xi_i' = \xi_i - 2\beta_{i1}\sum_{j=1}^{7}\beta_{j1}\xi_j \quad (i = 1, \cdots, 7).$$

Transforming $R_\beta C_1$ by O_{i234} for $i = 7, 6, 5$ successively, we may suppose that $\beta_{71} = \beta_{61} = \beta_{51} = 0$.

It is readily seen that a substitution R affecting only $\xi_1, \cdots, \xi_4$, is not commutative with every T_{ij} $(i, j = 1, \cdots, 5)$ for example not with T_{12}. Then I contains the substitution, not the identity,

$$R^{-1}T_{12}T_{67}RT_{67}T_{12} = T_\delta T_{12},$$

where $T_\delta \equiv R^{-1}T_{12}R$ has the form

$$T_\delta: \quad \xi_i' = \xi_i - \delta_i\sum_{j=1}^{4}\delta_j\xi_j \quad (i = 1 \cdots 4),$$

where $$\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = 2.$$

9. If $\delta_3 = \delta_4 = 0$, $T_\delta T_{12}$ becomes $Q^{\alpha,\beta}_{2,1}$ if we set

$$\alpha = \tfrac{1}{2}(\delta_1 - \delta_2), \quad \beta = \tfrac{1}{2}(\delta_1 + \delta_2).$$

Having $Q^{\alpha,\beta}_{2,1}$, I contains also $Q^{\alpha,\beta}_{2,3}$ and $Q^{\alpha,\beta}_{3,1}$ and thus the product of the two, which reduces to $T_\sigma T_{12}$, having

$$\sigma_1 = -1, \quad \sigma_2 = \alpha^2 - \beta^2, \quad \sigma_3 = 2\alpha\beta.$$

If $\alpha\cdot\beta = 0$, $Q^{\alpha,\beta}_{2,1} = C_1C_2$, not being the identity. Then, by § 7, I contains every $Q^{\alpha,\beta}_{i,j}$ and therefore, if $p^n > 5$, a substitution $Q^{\alpha,\beta}_{2,3}Q^{\alpha,\beta}_{3,1} = T_\sigma T_{12}$ in which $\alpha\beta \neq 0$.

10. Thus I contains a substitution $T_\delta T_{12}$ having δ_3 and δ_4 not both zero, say $\delta_4 \neq 0$. Transforming it by O_{246}, we can make the resulting substitution $T_{\delta'} T_{12}$ commutative with T_{12}. Indeed, the conditions

$$\delta'_6 \equiv \alpha\delta_3 + \beta\delta_4 = \delta_1, \quad \alpha^2 + \beta^2 + \gamma^2 = 1,$$

combine into the single condition

$$\alpha^2(\delta_3^2+\delta_4^2)-2\alpha\delta_1\delta_2+\delta_4^2\gamma^2=\delta_4^2-\delta_1^2.$$

For $\delta_3^2+\delta_4^2=0$, a solution is given by $\gamma=0$ when $\delta_1\neq 0$, and by $\gamma=1$ when $\delta_1=0$. For $\delta_3^2+\delta_4^2\neq 0$, there exist* solutions α, γ in the $GF[p^n]$ of the equivalent equation of condition

$$\{\alpha(\delta_3^2+\delta_4^2)-\delta_1\delta_3\}^2+\delta_4^2(\delta_3^2+\delta_4^2)\gamma^2=\delta_4^2(\delta_3^2+\delta_4^2-\delta_1^2).$$

Hence I contains the substitution, not the identity,

$$(T_{\delta'}\ T_{12})=T_{2\theta}\,T_{7\theta}\,(T_{\delta'}\ T_{12})^{-1}\,T_{7\theta}\,T_{2\theta}$$
$$=T_{\delta'}\ T_{1\theta}\,T_{\delta'}^{-1}\,T_{2\theta}=T_{1\theta}\,T_{2\theta}.$$

The alternating group on $m>4$ letters being simple, the group I, containing $T_{1\theta}\ T_{2\theta}$, contains the whole alternating group. Further, $C_1\,C_2$ transforms $T_{1\theta}\,T_{2\theta}$ into $T_{1\theta}\,T_{2\theta}\,C_1\,C^2$, so that I contains $C_1\,C_2$ and therefore every $C_i\,C_j$. Hence, by § 7, I contains every $Q_{i,i}^{\alpha,\beta}$. Thus the group I coincides with H.

Addenda† of April 18.

11. For $p^n=3$ or 5, the maximal invariant sub-group of H is of order 2 or 1, according as $m>4$ is even or odd. For $p^n=3$, $m=4$, the order of H is $2^5\cdot 3^2$, and its factors of composition are all primes.

12. Suppose $p^n=8l\pm 1$, so that 2 is a square. Let $O_{i,2}^{\alpha,\beta}$ denote a definite orthogonal substitution not in $Q_{1,2}$, so that $1\pm\alpha$ are not-squares. Denote by H_1 the group obtained by extending the group of the $Q_{i,j}$ by all the products $O_{i,j}^{\alpha,3}\,O_{i,l}^{\alpha,\beta}$.

Theorem: *H_1 contains half of the substitutions of G.* For every substitution of G is of the form

$$S=h_1\,(O_{i,j}^{\alpha,\beta}\,h_2\,O_{k,l}^{\alpha,\beta}\cdots,$$

$h_1, h_2, \cdots, h$ denoting substitutions of H_1. Now $O_{i,j}^{\alpha,\beta}$ can be carried to the right of every $Q_{i,j}^{\lambda,\mu}$ and every $Q_{k,l}^{\lambda,\mu}$ $(k,l\neq i,j)$. Further, since $(O_{1,2}^{\alpha,\beta})^2=Q_{1,2}^{\alpha,-\beta}$,

$$O_{i,j}^{\alpha,\beta}\,Q_{i,k}^{\lambda,\mu}=O_{i,j}^{\alpha,\beta}\,(O_{i,k}^{\alpha,\beta})^2\,Q_{i,k}^{\alpha,\beta}\cdot Q_{i,k}^{\lambda,\mu},$$
$$=O_{i,j}^{\alpha,\beta}\,O_{i,k}^{\alpha,\beta}\cdot Q_{i,k}^{\alpha,\beta}\,Q_{i,k}^{\lambda,\mu}\cdot O_{i,k}^{\alpha,\beta}=h\,O_{i,k}^{\alpha,\beta}.$$

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* Bulletin, l. c. § 3.

† The results here announced will be proven in full in the *Proceedings of the California Academy of Sciences*, Third Series, vol. 1, No. 5.

Thus S finally takes the form

$$k'\ O_{r,s}^{\alpha,\beta} = k'\ O_{r,s}^{\alpha,\beta}\ O_{2,1}^{\alpha,\beta}\ O_{1,2}^{\alpha,\beta} = k''\ O_{1,2}^{\alpha,\beta}.$$

13. THEOREM: *For $m > 4$, $p+2$, the maximal invariant sub group of H_1 is of order 2 or 1 according as m is even or odd.*

For $m > 7$, the group is similar to that for the group H as given above. In §6 we replace $T_{12}\ C_1\ O_{345}$ by $O_{1,2}^{\alpha,\beta}\ O_{345}$. We replace §7 by the

Lemma: If $p^n = 8l \pm 1$, $m > 3$, an invariant sub-group I containing every C_iC_j coincides with H_1.

Indeed, $O_{2,4}^{\alpha,\beta}\ O_{1,2}^{\alpha,\beta}$ transforms $C_1\ C_3$ into $Q_{1,2}^{\alpha,\beta}\ C_1\ C_3$, so that I contains every $Q_{i,j}^{\alpha,\beta}$. Having $T_{23}C_3$, I contains every $O_{i,j}^{\alpha,\beta}\ O_{k,l}^{\alpha,\beta}$. Thus, for example,

$$(T_{23}C_3)\,(O_{1,2}^{\alpha,\beta}\ O_{1,4}^{\alpha,\beta})\,(T_{23}C_3)^{-1}\,(O_{1,2}^{\alpha,\beta}\ O_{1,4}^{\alpha,\beta})^{-1} = O_{1,3}^{\alpha,\beta}\ O_{2,1}^{\alpha,\beta}.$$

14. THEOREM: *For $p > 2$, the ternary orthogonal group in the $GF[p^n]$ has a sub-group H' of index two and of order $\frac{1}{2}p^n(p^{2n}-1)$ which is simply isomorphic to the group of linear fractional substitutions of determinant unity on a single index.*

Indeed, the orthogonal substitution

$$S\colon\ \xi_i' = \sum_{j=1}^{3} a_{ij}\xi_j \quad (i = 1, 2, 3),$$

expressed in terms of the new indices

$$\eta_1 = -i\xi_1, \quad \eta_2 = \xi_2 - i\xi_3, \quad \eta_3 = \xi_2 + i\xi_3,$$

leaves $\eta_1^2 - \eta_2\eta_3$ invariant and has the form

$$S_1\colon \begin{Bmatrix} a_{11} & \frac{1}{2}(a_{13}-ia_{12}) & -\frac{1}{2}(a_{13}+ia_{12}) \\ a_{31}+ia_{21} & \frac{1}{2}(a_{33}-ia_{32}+ia_{23}+a_{33}) & \frac{1}{2}(a_{33}-ia_{32}-ia_{23}-a_{33}) \\ -a_{31}-ia_{21} & \frac{1}{2}(a_{33}+ia_{32}+ia_{23}-a_{33}) & \frac{1}{2}(a_{13}+ia_{32}-ia_{23}+a_{33}) \end{Bmatrix}$$

Understanding by H' the group H or H_1 according as $p^n = 8l \pm 3$ or $p^n = 8l \pm 1$, we may verify that for every substitution of H' the coefficient $\frac{1}{2}(a_{22} - ia_{32} + ia_{23} + a_{33})$ is the square of a complex α of the form $\rho + \sigma i$, where ρ and σ are marks of the $GF[p^n]$. It readily follows* that S_1 may be written in the form:

$$S_1\colon \begin{Bmatrix} \alpha\delta + \beta\gamma & \alpha\gamma & \beta\delta \\ 2\alpha\beta & \alpha^2 & \beta^2 \\ 2\gamma\delta & \gamma^2 & \delta^2 \end{Bmatrix} \qquad [\alpha\delta - \beta\gamma = 1]$$

where α is conjugate to δ, β to γ. Further two such ternary

*Compare Klein-Fricke: Automorphic Functions I., p. 14; Weber: Algebra, II., p. 190.

substitutions have the same composition formula as linear fractional substitutions. Hence, according as -1 is a square or a not-square, H' is simply isomorphic to the "real" or the "imaginary" form* of the group of linear fractional substitutions of determinant unity. Thus, for $p^n > 3$, H' is simple.

15. Observing that the squares of the substitutions

$$O_{1,2}^{\alpha,\beta}, \quad O_{1,2}^{\alpha,\beta}\, T_{13} C_1 C_2 C_3, \quad O_{1,2}^{\alpha,\beta}\, T_{13} T_{24}$$

are respectively $Q_{1,2}^{\alpha,-\beta}{}_{,\gamma}$, $O_{1,2}^{\alpha,\beta}\, O_{3,4}^{\alpha,\beta}$, $O_{1,2}^{\alpha,\beta}\, O_{3,4}^{\alpha,\beta}$, we may unite our results into the following

THEOREM: *The squares of the linear substitutions on m indices in the $GF[p^n]$, $p \neq 2$, which leave invariant the sum of the squares of the m indices, generate a group, which for $m = 2k+1$ has the order*

$$\tfrac{1}{2}(p^{2nk}-1)\, p^{2nk-n}\, (p^{2nk-2n}-1)\, p^{2nk-3n} \cdots (p^{2n}-1)\, p^n$$

and is simple except when $p^n = 3$, $m = 3$; while for $m = 2k > 4$ it has the factors of composition 2 and

$$\tfrac{1}{4}[p^{nk} - (\pm 1)^k]\, p^{nk-n}\, (p^{2nk-2n}-1)\, p^{2nk-2n} \cdots (p^{2n}-1)\, p^n,$$

the sign $\pm$ depending upon the form $4l \pm 1$ of p^n.

UNIVERSITY OF CALIFORNIA,
February 10, 1898.

A PROOF OF THE THEOREM:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

BY MR. J. K. WHITTEMORE.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

THEOREM: *Let $u = f(x, y)$ denote a function of the two independent variables x and y which, together with its first derivatives and the two second derivatives in question, is continuous in the neighborhood of the point (x, y); then* $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

Let $\dfrac{\partial^2 f(x,y)}{\partial x \partial y}$ denote $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f(x,y)}{\partial y}\right)$

* Moore: Mathematical Papers of the Chicago Congress (1893), "A doubly-infinite system of simple groups," §§ 5–6.

and let $\dfrac{\partial^2 f(x,y)}{\partial y \partial x}$ denote $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f(x,y)}{\partial x}\right)$.

Let (x_0, y_0) be any point for which the conditions of the theorem are fulfilled and let the lines $x = a$, $x = b$, $y = c$, $y = d$ bound a region of the plane enclosing the point (x_0, y_0) and so small that the conditions stated are satisfied throughout the interior of the rectangle and on its boundary. Under these conditions we have

$$\int_c^d dy \int_a^b dx \frac{\partial^2 f(x,y)}{\partial x \partial y} = f(b,d) - f(b,c) - f(a,d) + f(a,c),$$

$$\int_a^b dx \int_c^d dy \frac{\partial^2 f(x,y)}{\partial y \partial x} = f(b,d) - f(a,d) - f(b,c) + f(a,c).$$

But, under the conditions of the theorem,

$$\int_a^b dx \int_c^d dy \frac{\partial^2 f(x,y)}{\partial y \partial x} = \int_c^d dy \int_a^b dx \frac{\partial^2 f(x,y)}{\partial y \partial x}.$$

Hence $$\int_c^d dy \int_a^b dx \left(\frac{\partial^2 f(x,y)}{\partial x \partial y} - \frac{\partial^2 f(x,y)}{\partial y \partial x}\right) = 0.$$

Now, if a function, continuous in the neighborhood of a point (x_0, y_0), is such that its integral, extended over *any* rectangle enclosing this point, is zero, it is readily seen that the function cannot be positive or negative at the point (x_0, y_0). Hence

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} - \frac{\partial^2 f(x,y)}{\partial y \partial x} = 0$$

at the point (x_0, y_0). But this was any point, and the theorem is proved.

SOME OBSERVATIONS ON THE MODERN THEORY OF POINT GROUPS.

BY MISS FRANCES HARDCASTLE.

THE origins of the theory of point groups are to be found in Brill and Noether's classic memoir (see infra) published nearly twenty-five years ago, but it is only within the last fifteen years that systematic attention has been given to the subject by the Italian mathematicians, Segre, Bertini,

Castelnuovo and others; in their hands a series of isolated problems bids fair to develop into an organic theory, a theory, moreover, which furnishes links in thought between subjects as apparently diverse as projective geometry and the analytical theory of elimination, while maintaining the initial, direct connection between the theory of higher plane curves and the theory of functions. In the following pages I have briefly indicated some of the converging lines of the German and Italian work; in the first section, by a discussion of certain of the technical terms; in the second section, starting from the so-called Riemann-Roch equations, by the suggestion of certain lines of enquiry which may prove useful in the classification of algebraic curves. The appended bibliography is divided into two parts; the first contains those memoirs which may be considered as fundamental, from the historical as well as from the purely technical point of view, to the theory of point groups, including those more especially concerned with linear systems of *plane* curves; the second includes memoirs on the theory of curves in ordinary and in hyperspace, more or less directly connected with the theory of point groups.

§1.

The geometry of a space of a given number of dimensions treats of the characteristic properties of the configurations existing in that space and formed by spaces of a smaller number of dimensions. The geometry on a plane curve, a one-dimensional space, deals, therefore, with configurations formed of points and studies the laws which they follow. In this connection the curve is called the *base curve*, the configurations are *point groups*. A point group which presents itself very naturally is that formed by the intersections of the base curve with any other specified plane curve, and among such special attention has been directed to those obtained by means of the so-called *adjoint curves*—curves, namely, which have an $(i-1)$-fold point at every i-fold point of the base curve (whether satisfying further conditions or not). In the discussions of such point groups, those intersections of the adjoint curve with the base curve which necessarily coincide with the i-fold points of the latter* are excluded. A point group in this portion of the theory is composed of some or all of the remaining points of intersec-

* The i-fold points of the base curve are often referred to as the *base points*, the remaining points of intersection of an adjoint curve with the base curve as the *moveable points of intersection*.

tion of an adjoint curve and the base curve. Of fundamental importance in these investigations is the *Theorem of Residuation*, which introduces the notions of *residual* and *coresidual point groups* and establishes a characteristic property of such point groups. The definitions are as follows: That point group G_Q of Q points, which together with a point group G_R of R points, makes up the whole number of intersections (always excluding the base points) of the base curve with an adjoint curve of arbitrarily assumed order, is said to be *residual to* G_R and vice versa; two point groups G_R, $G_{R'}$ are said to be *coresidual with respect to a third point group* G_Q when each is residual to G_Q; and it is noteworthy that this definition applies whether the orders of the adjoint curves which pass through the pairs of point groups G_Q, G_R and G_Q, $G_{R'}$ respectively, be the same or different; thus the coresidual point groups G_R, $G_{R'}$ do not necessarily contain the same number of points. The theorem in its simplest form is thus worded: *if the point groups* G_R, $G_{R'}$, $\cdots$ *are coresidual with respect to a point group* G_Q, *they are also coresidual with respect to every other point group* $G_{Q'}$, *residual to any one of their number.**

The proof of the theorem of residuation is such that it is capable of immediate extension by introducing, in the place of a single point group, the wider conception of a *system of point groups*, determined on the base curve by a system of adjoint curves. A system of point groups, namely, G_R, $G_{R'}$, $\cdots$, is determined on the base curve by means of a system of adjoint curves, in which every curve is constrained to pass through a given point group G_Q, (G_R, $G_{R'}$, $\cdots$ form, in fact, a coresidual system in which $R = R' = \cdots$, numerically); or, we may say, this system of adjoint curves is determined by G_Q and determines G_R, $G_{R'}$, $\cdots$. The theorem of residuation then shows that, *if* $G_{Q'}$ *be any point group coresidual to* G_Q *with respect to* G_R, *the system of adjoint curves determined by* G_Q *is equivalent to that determined by* $G_{Q'}$, *the same system of point groups* G_R, $G_{R'}$, $\cdots$ *being determined on the base curve by either system of adjoint curves.* The equation of a system of adjoint curves involves one or more variable parameters, which may, without invalidating the truth of the theorem, enter in any specified degree; the most simple case, that of *linear systems*, however, is the only one which

*The nomenclature is due to Sylvester (cf. Salmon's Higher Plane Curves, 2d edit. (1873), p. 131), who proved the theorem for cubics; the notation, G_Q, was introduced by Brill and Noether in the memoir referred to above, viz.: "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie," *Math. Annalen.*, vol. 7 (1873).

has been at all fully worked out. And the importance of the theorem of residuation lies in the fact that the number r of linearly independent curves of a linear system of adjoint curves determined by *any* one of a coresidual system of point groups, G_Q, $G_{Q'}$, ... is the same; this number r is therefore characteristic of a unique coresidual system of point groups, G_R, $G_{R'}$, ... and may be affixed as an index to the symbol denoting any one of these point groups; *e. g.*, G^r_R. If it is not essential to fix the attention on any one point group of such a coresidual system, but rather on the linear system itself (of which the single point group is an element) *the symbol g^r_R is employed.*

This is the original, limited meaning of this symbol as introduced by Brill and Noether (*loc. cit.*). Later, in the Italian memoirs, the symbol is used to denote *any* system of point groups on a base curve, whether determined by adjoint curves or not and whether these adjoint curves, if employed, satisfy further conditions or not; it simply means that the system of point groups is r-ply infinite and that each element is composed of a group of R points on the base curve; *the order of the system g^r_R is R, the dimensions r.* And then, if g^r_R be a *linear* system (whether determined by adjoint curves or not) a distinction is drawn between "*complete*" *systems* (serie completa, Vollschaar) and "*partial*" *systems* (serie parziale, Teilschaar),* viz., a linear system is complete if it is not contained in another of the *same* order but of *higher* dimensions; partial, if it is so contained.

With reference to the relation between the values of R and r in a linear g^r_R, it can be proved† that, the deficiency of the base curve being a known quantity p, a g^r_R for which $R - r \geq p - 1$ can always be determined on the base curve by means of a system of adjoint curves of order $n - 3$ (where n is the order of the base curve); and this holds whether R be less than or equal to the whole number of intersections (excluding the base points) of a curve of the system with the base curve; such systems of point groups are said to be *special systems.* It can also be proved that the linear system determined on the base curve by a system of adjoint curves of any order which satisfies no further condition than those due to its passage through the base points (and possibly through ordinary points on the base curve) is always complete;‡ thus a special system *if* determined by such a system of adjoint curves (of order $n - 3$) is com-

* Cf. Segre, *Annali di matematica*, vol. 22, p. 65, (1894).
† Brill and Noether, *loc. cit.*, p. 278.
‡ Cf. Bertini, *Annali di matematica*, vol. 22, p. 9, (1894).

plete, but need not necessarily be so. Moreover it follows* from the theorem of residuation and from the nature of a special system that there are exactly p linearly independent adjoint curves of order $n-3$; each of these cuts the base curve in $2p-2$ points. In other words, then, there exists on the base curve a g^{p-1}_{2p-2} which is *special* and *complete;* this system has been termed the *canonical system* (serie canonica).†

The above definition of a special system is not that originally given by Brill and Noether; they defined‡ a special point group G'_R as one for which

$$r > R-p+1, \text{ i. e., } R-r<p-1,$$

thus omitting the case in which $R-r=p-1$. Lindemann has adopted this original definition in his edition of Clebsch's Geometrie§, published in 1876, but in Brill and Noether's report on the theory of algebraic functions‖, published in 1894, the wider definition is adopted and the distinction between the two cases is effected by the introduction of the term *extraordinary point group* (ausgezeichnete Gruppe) to designate any point group which presents *fewer* conditions to a given system of curves (whether adjoint curves, or not) than the number of points it contains; a point group for which $R-r<p-1$, is *extraordinary with respect to the linear system of adjoint curves of order* $n-3$, it is a particular case of an extraordinary point group, being, in fact, an *extraordinary, special* point group; a point group for which $R-r=p-1$, on the other hand, is *special* but not extraordinary. The Italian mathematicians use the wider definition, sometimes in the form: g'_R *is a special point group if* $R-r \gtrless p-1$ ¶; sometimes in the equivalent form: g'_R *is a special point group if it can be determined on the base curve by a linear system of adjoint curves of order* $n-3$**. Moreover, instead of marking out a given point group as extraordinary *with respect to a given system of curves*, they say that a given system of curves is *regular* (regolare)†† *with respect to a given point group* when the conditions which every curve of the system must satisfy in order to pass through this point group are all independent, and if the system is *not* regular, *i. e.*, if the conditions

* Brill and Noether, *loc. cit.*, p. 280. Cf. also Bertini, *loc. cit.*, p. 17.

† Segre, *loc. cit.*, p. 118.

‡ *loc. cit.* p. 280.

§ p. 699.

‖ *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 3, pp. 542, 543.

¶ Segre, *loc. cit.*, p. 15.

** Bertini, *loc. cit.*, p. 111.

†† Castelnuovo, *Mem. d. Reale Accad. di Torino*, vol. 42, (1891).

are not all independent (in the German terminology, *if the given point group is extraordinary with respect to the given system of curves*) the number of conditions which follow from the rest is called the *excess* (sovrabbondanza)* *of the system of curves.*

In Brill and Noether's paper, the main object of which was the algebraic formulation of theorems known to be true from the theory of functions, point groups, as such, only present themselves incidentally and attention is almost exclusively directed to the special systems, and, moreover, to complete special systems. It was there shown for the first time that complete special systems of point groups always present themselves in pairs in such a manner that the existence of one system carries with it the existence of the complementary system. Their statement of this theorem is as follows: *if through a point group G_Q^q on the base curve where $q = Q - p + 1 + r$ (r being a positive integer less than $p - 1$) an adjoint curve of order $n - 3$ be passed, it will intersect the base curve in $2p - 2 - Q = R$ more points which again belong to a special system g_R^r, the value of r being $R - p + 1 + q$ derived from the equation above.* Brill and Noether call this theorem the *Riemann-Roch Theorem* but Klein points out that the *Brill-Noether Theorem of Reciprocity* would be the more appropriate title. The equations involved may be written

$$Q + R = 2p - 2, \qquad (i)$$
$$Q - R = 2(q - r), \qquad (ii)$$

and are usually referred to in this form.†

§ 2.

Two problems present themselves with reference to complete special systems of point groups regarded in connection with equations (*i*) and (*ii*).

(*a*) On a given base curve to enumerate all possible combinations of values of Q, q—whence R, r are known—and to show how each such g_Q^q is determined.

* Castelnuovo, *Ibid.*, p. 9. F. S. Macaulay in a paper on "Point Groups in Relation to Curves" (*Lond. Math. Soc. Proc.*, vol. 26, pp. 495–544, 1895), defined *excess* with reference to the point group not to the system of curves (p. 499) but, with care, no ambiguity need arise from this: Professor Klein in his lectures on Riemann's surfaces delivered in Göttingen, in 1891–92, employed the term *Ueberschuss*; it is used with reference to the *point group*, but turns out to be *greater by unity than the excess*, owing to the way in which it is originally defined (see lithographed lectures on Riemann'sche Flächen, I., p. 108; II., pp. 76, 77).

† *loc. cit.*, I, p. 108.

(β) To find on which base curves any given special system g^r_q—and hence also its known complementary system g^s_R—can be determined and how.

By saying that a base curve is *given* is meant, not only that n and p are known, but also what particular combination of double points and other (ordinary) higher singularities produce the given value of p. And to "show how" means to decide the geometrical laws which govern every point group of the g^r_q, *e. g.*, that 3 points of every G^r_q lie on a straight line, that 2 points of every G^r_q and a singularity of the base curve lie on a straight line, etc.

In the first place, it is necessary to establish a limitation to the possible combinations of values of Q and q, since equations (*i*) and (*ii*), for a given value of p, admit of an infinite number of such combinations. The fact that the questions involved in a one-one transformation of the base curve were of primary importance in Brill and Noether's paper led them in the second part of their memoir to deal exclusively with those base curves which are said to be "general" for a given deficiency. And the necessary limitation on the values of Q and q for a special system g^r_q on such a base curve is found to be $p-(q+1)(r+1)\geq 0$. But if we regard the problem as formulated in (α) we do not necessarily confine our attention to "general" base curves and a limitation to the values of Q, q may exist, which, *while including all combinations permitted by Brill and Noether's inequality, permit others which theirs excludes.* Such a limitation was first established in the theory of abelian functions by Clifford,* if $Q\geq 2q$. It can be found without reference to abelian functions as follows:†

The Theorem of Reciprocity shows that by constraining the $(p-1)$-ply infinite system of adjoint curves of order $n-3$ to pass through the Q points of any G^r_q belonging to a given g^r_q, we determine on the base curve a g^s_R, and that by constraining the $(p-1)$-ply infinite system of adjoint curves of order $n-3$ to pass through the R points of any G^s_R of this g^s_R we determine on the base curve the same g^r_q initially given. If we add the degrees of freedom of the element G^r_q to the degrees of freedom of the system g^s_R determined by it, the sum cannot *exceed* the number of degrees of freedom of the system of adjoint curves of order $n-3$ by means of which both the g^r_q and the g^s_R are determined, *i. e.*, $q+r\leq p-1$. And by equations (*i*) and (*ii*) this is

* "On the Classification of Loci," *Phil. Trans.*, 1878. (Math. Papers, p. 329.)

† Cf. also Bertini: *Annali di matematica*, vol. 22, p. 20.

equivalent to $Q-2q\geq 0$. Now, since $Q=p-1+q-r$, $Q-2q=p-(q+1)(r+1)+qr$ and hence Clifford's theorem permits all combinations of values of Q and q included in Brill and Noether's formula and also permits other combinations for which $p-(q+1)(r+1)$ is negative, provided this quantity is numerically less than qr. For instance, if $p=6$, $g_3^1 g_7^3$ makes $p-(q+1)(r+1)$ negative, but this pair of special systems can be found on a sextic *with a triple point and a double point*, for it is determined by the degenerate system of adjoint cubics consisting of a straight line through the triple point, cutting the base curve in a G_3' and a conic through the triple point and the double point cutting the base curve in a G_7^3. Again, if $p=7$, $g_4^1 g_8^3$ makes $p-(q+1)(r+1)$ negative, but, nevertheless, exists on a sextic with 3 double points, being determined by a straight line through one double point and a conic through the remaining two.

This restriction, $Q\geq 2q$, applies to every base curve, but on a *given* base curve, it may be possible to find further restrictions of a similar character, viz., $Q\geq xq$, where x has some value other than 2. There is, however, a unique character about the value $x=2$, for this reason:

let $$Q=xq+k,\ (k\gtreqless 0);$$

then since $$r=p-1+q-Q,$$

$$R-xr=2p-2-Q-x(p-1+q-Q)$$

$$=(2-x)(p-1-Q)+k.$$

Thus, if $x=2$, $R\gtreqless xr$ according as $Q\gtreqless xq$, but if $x>2$, this is only true when $Q=p-1$; if $x>2$ and $Q<p-1$, R *must* be $<xr$, *if* $Q\leq xq$; the other possible cases, however, give no *necessary* limits to the values of R dependent on the values of Q.

As a particular example of such a restriction on a particular base curve, it can be shown* that no pair of complementary special systems g_0^r, g_n^r *for which* $Q<3q$ *and* $R<3r$ can exist on a curve of the nth degree with an $(n-3)$-fold point, T triple points and $2n-5-p-3T$ double points.

* Hardcastle: *Lond. Math. Soc. Proc.*, vol. 29.

BIBLIOGRAPHY.

I.

1857. RIEMANN: Theorie der Abelschen Functionen. *Crelle*, vol. 54, pp. 115–155.
1864. ROCH: Ueber die Anzahl der willkürlichen Constanten in algebraischen Functionen. *Crelle*, vol. 64, pp. 372–376.
1866. CLEBSCH and GORDAN: Theorie der Abelschen Functionen.
1873. BRILL and NOETHER: Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie. *Mathematische Annalen*, vol. 7, pp. 269–310.
1874. CAYLEY: On the group of points G_4^1 on a sextic curve with five double points. *Mathematische Annalen*, vol. 8, pp. 359–362. Collected Works, IX, p. 504.
1878. CLIFFORD: On the classification of loci. *Philosophical Transactions*, vol. 169, part 2, pp. 663–681 (Mathematical papers, XXXIII., pp. 305–331).
1879. NOETHER: Ueber die Schnittpunktsysteme einer algebraischen Curve mit nicht-adjungirten Curven. *Mathematische Annalen*, vol. 15, pp. 507–528.
1879. KRAUS: Note über aussergewöhnliche Specialgruppen auf algebraischen Curven. *Mathematische Annalen*, vol. 16, pp. 245–259.
1881. CAPORALI: Sopra i sistemi lineari. Collectanea mathematica (edid. Cremona and Beltrami).
1882. BERTINI: Sui sistemi lineari. *Rendiconti Lombardi*, vol. 15, pp. 24–28.
1883. NOETHER: Rationale Ausführung der Operationen in der Theorie der algebraischen Functionen. *Mathematische Annalen*, vol. 23, pp. 310–358.
1886. GUCCIA: Generalizzazione di un teorema di Noether. *Rendiconti di Palermo*, vol. 1, pp. 139–156.
1886. NOETHER: Ueber die reductiblen algebraischen Curven. *Acta Mathematica*, vol. 8, pp. 161–192.
1886. GUCCIA: Sulle superficie algebriche le cui sezioni piane sono unicursali. *Rendiconti di Palermo*, vol. 1, pp. 165–168.
1886. GUCCIA: Sulla riduzione dei sistemi lineari delle curve ellittiche, e sopra un teorema generale delle curve algebriche di genere p. *Rendiconti di Palermo*, vol. 1, pp. 169–189.
1886. BOBEK: Ueber hyperelliptische Curven. *Mathematische Annalen*, vol. 29, pp. 386–412.
1887. MARTINETTI: Sopra una classe di sistemi lineari di curve piane algebriche. *Rendiconti di Palermo*, vol. 1, pp. 202–204.
1887. MARTINETTI: Sopra alcuni sistemi lineari di curve piane algebriche di genere due. *Rendiconti di Palermo*, vol. 1, pp. 205–216.
1887. SEGRE: Sui sistemi lineari di curve piane algebriche di genere p. *Rendiconti di Palermo*, vol. 1, pp. 217–221.

1887. JUNG: Ricerche sui sistemi lineari di curve algebriche di genere qualunque. *Annali di matematica*, vol. 15, pp. 277–312.

1887. MARTINETTI: Sui sistemi lineari di genere 1. *Rendiconti Lombardi*, vol. 20, pp. 264–268.

1888. JUNG: Ricerche sui sistemi lineari di genere qualunque e sulla loro riduzione all' ordine minimo. *Annali di matematica*, vol. 16, pp. 291–326.

1888. KÜPPER: Ueber die auf einer Curve $m$ter Ordnung C_p^m vom Geschlecht p von den ∞^1 Geraden G der Ebeneausgeschnittene lineare Schaar g_m^2. *Mathematische Annalen*, vol. 31, pp. 291–301.

1888. BERTINI: Sulle curve fondamentali dei sistemi lineari di curve piane algebriche. *Rendiconti di Palermo*, vol. 3, pp. 3–21.

1888. CASTELNUOVO: Geometria sulle curve ellittiche. *Atti di Torino*, vol. 24, pp. 4–22.

1889. CASTELNUOVO: Ricerche di geometria sulle curve algebriche. *Atti di Torino*, vol. 24, pp. 346–373.

1889. CASTELNUOVO: Numero delle involuzioni razionali giacenti sopra una curva di dato genere. *Rendiconti dei Lincei*, 5th ser. vol. 2, pp. 130–132.

1889. BOBEK: Ueber Dreischaarcurven. *Wiener Berichte*, vol. 98, pp. 142–173.

1889. KÜPPER: Ueber die Curven C_p^n von $n$ter Ordnung und dem Geschlecht $p > 1$ auf welchen die einfachsten Specialschaaren g_2^1, g_3^1 vorkommen. *Prager Abhandlungen*, 7th ser., vol. 3, 18 pp.

1890. KÜPPER: Zur Theorie der algebraischen Curven $n$ter Ordnung C^n. *Prager Abhandlungen*, 7th ser., vol. 3, 25 pp.

1890. JUNG: Un' osservazione sul grado massimo dei sistemi lineari di curve piane algebriche. *Annali di matematica*, vol. 18, pp. 129–130.

1890. CASTELNUOVO: Massima dimensione dei sistemi lineari di curve piane di dato genere. *Annali di matematica*, vol. 18, pp. 118–128.

1890. JUNG: Delle famiglie associate di sistemi lineari e delle superficie univocamente rappresentabili sul piano. *Rendiconti di Palermo*, vol. 4, pp. 253–260.

1890. BERTINI: Intorno ad alcuni teoremi della geometria sopra una curva algebrica. *Atti di Torino*, vol. 26, pp. 117–130.

1890. BRILL: Ueber algebraische Correspondenzen. Zweite Abhandlung: Specialgruppen von Punkten einer algebraischen Curve. *Mathematische Annalen*, vol. 36, pp. 321–360.

1891. CASTELNUOVO: Alcune osservazioni sopra le serie irrazionali di gruppi di punti appartenenti ad una curva algebrica. *Rendiconti dei Lincei*, 4th ser., vol. 7, pp. 294–299.

1891. CASTELNUOVO: Ricerche generali sopra i sistemi lineari di curve piane. *Memorie di Torino*, vol. 42, pp. 3–43.

1892. AMODEO: Còntribuzione alla teoria delle serie irrazionali involutorie ∞^1 giacenti sulle varietà algebriche ad una dimensione. *Annali di matematica*, vol. 20, pp. 227–235.

1893. CASTELNUOVO: Sulla linearità delle involuzioni più volte infinite appartenenti ad una curva algebrica. *Atti di Torino*, vol. 28, pp. 727–738.

1893. AMODEO: Curve k-gonali. *Annali di matematica*, vol. 21, pp. 221–256.

1893. AMODEO: Curve aggiunte minime. *Rendiconti dei Lincei*, 5th ser. vol. 2, pp. 460–467.

1893. AMODEO: Serie residua nella serie canonica delle curve aggiunte di ordine $m-3-a$. *Rendiconti dei Lincei*, 5th ser., vol. 2, pp. 528–532.

1894. BERTINI: La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico. *Annali di matematica*, vol. 22, pp. 1–40.

1894. SEGRE: Introduzione alla geometria sopra un ente algebrico semplicemente infinito. *Annali di matematica*, vol. 22, pp. 41–142.

1895. KÜPPER: Ueber k-gonale Curven C_p^n nter Ordnung, vom Geschlecht p. *Prager Sitzungsberichte*, 16 pp.

1895. KÜPPER: Zur Theorie der algebraischen Curven. *Monatshefte*, vol. 6, pp. 127–157.

1895. CIANI: Sopra i sistemi lineari di curve algebriche piane. *Giornale di matematiche di Battaglini*, vol. 33, pp. 57–79.

1896. KÜPPER: Nachtrag zu den k-gonalen Curven. *Prager Sitzungsberichte*, 9 pp.

1896. AMODEO: Curve k-gonali di 1a e di 2a specie. *Annali di matematica*, vol. 24, pp. 1–22.

II.

1862. CAYLEY: Considérations générales sur les courbes en espace. *Comptes rendus*, vol. 54, pp. 55–60, 396–401, 672–678. Collected Works, vol. V., p. 7.

1870. NOETHER: Ueber Flächen, welche Schaaren rationaler Curven besitzen. *Mathematische Annalen*, vol. 3, pp. 161–227.

1874. NOETHER: Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde. *Mathematische Annalen*, vol. 8, pp. 495–532.

1878. CLIFFORD: On the classification of loci. *Philosophical Transactions*, vol. 169, part 2, pp. 663–681. Mathematical Papers, p. 329.

1881. VERONESE: Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens. *Mathematische Annalen*, vol. 19, pp. 161–234.

1882. NOETHER: Zur Grundlegung der Theorie der algebraischen Raumcurven. *Crelle*, vol. 93, pp. 271–318.

1883. MOLK : Sur une notion qui comprend celle de la divisibilité et sur la théorie générale de l'élimination. *Acta Mathematica*, vol. 6, pp. 1–166.
1885. DEL PEZZO : Sulle superficie di ordine n immerse nello spazio di $n+1$ dimensioni. *Rendiconti di Napoli*, vol. 24, pp. 212–216.
1885. SEGRE: Sulle varietà normali à tre dimensioni, composte di serie semplici razionali di piane. *Atti di Torino*, vol. 21, pp. 95–115.
1886. SEGRE : Ricerche sulle rigate ellittiche di qualunque ordine. *Atti di Torino*, vol. 21, pp. 868–891.
1886. DEL PEZZO : Sulle projezioni di una superficie e di una varietà dello spazio ad n dimensioni. *Rendiconti di Napoli*, vol. 25, pp. 205–213.
1887. GUCCIA : Sui sistemi lineari di superficie algebriche dotate di singolarità à base qualunque. *Rendiconti di Palermo*, vol. 1, pp. 338–349.
1887. DEL PEZZO : Sulle superficie dell' $n$mo ordine immerse nello spazio di n dimensioni. *Rendiconti di Palermo*, vol. 1, pp. 241–271.
1887. SEGRE : Intorno alla geometria su una rigata algebrica. *Rendiconti dei Lincei*, 4th ser., vol. 3, pp. 3–6.
1887. SEGRE : Sulle varietà algebriche composte di una serie semplicemente infinita di spazi. *Rendiconti dei Lincei*, 4th ser., vol. 3, pp. 149–153.
1887. SEGRE : Recherches générales sur les courbes et les surfaces réglées algébriques. Part I. *Mathematische Annalen*, vol. 30, pp. 203–226.
1888. SEGRE : Sulle curve normali di genere p dei vari spazi. *Rendiconti Lombardi*, 2d ser., vol. 21, pp. 523–528.
1888. CASTELNUOVO : Una applicazione della geometria enumerativa. *Rendiconti di Palermo*, vol. 3, pp. 27–37.
1889. CASTELNUOVO : Numero degli spazi che segano più retti in uno spazio di n dimensioni. *Rendiconti dei Lincei*, 4th ser., vol. 5, pp. 71–78.
1889. SEGRE : Recherches générales sur les courbes et les surfaces réglées algébriques. Part II. *Mathematische Annalen*, vol. 34, pp. 1–25.
1889. DEL PEZZO : Sui sistemi di curve e di superficie. *Rendiconti di Palermo*, vol. 3, pp. 236–240.
1890. CASTELNUOVO : Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche. *Rendiconti di Palermo*, vol. 4, pp. 73–88.
1890. CASTELNUOVO : Sulle superficie le cui sezioni sono curve di genere 3. *Atti di Torino*, vol. 25, pp. 695–715.
1893. CASTELNUOVO : Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica. *Rendiconti di Palermo*, vol. 7, pp. 89–111.
1893. CASTELNUOVO : Sulla razionalità delle involuzioni piane. *Rendiconti dei Lincei*, 5th ser., vol. 2, pp. 205–209.
1893. ENRIQUES : Una questione sulla linearità dei sistemi di

curve appartenenti ad una superficie algebrica. *Rendiconti dei Lincei*, 5th ser., vol. 2, pp. 3–8.

1893. ENRIQUES: Sui sistemi lineari di superficie algebriche le cui intersezioni variabili sono curve iperellittiche. *Rendiconti dei Lincei*, 5th ser., vol. 2, pp. 281–287.

1896. ENRIQUES: Sopra le superficie algebriche di cui le curve canoniche sono iperellittiche. *Rendiconti dei Lincei*, 5th ser., vol. 5, pp. 191–197.

NOTE ON CONTACT TRANSFORMATIONS.

BY PROFESSOR EDGAR ODELL LOVETT.

IN a paper "Ueber die geodätische Krümmung der auf einer Fläche gezogenen Curven und ihre Aenderung bei beliebiger Transformation der Fläche" in the Zeitschrift für Mathematik und Physik, Vol. XXXVII., 1892, Dr. Mehmke, Schlömilch's successor as one of the editors of that journal, derives some interesting theorems relative to the change in the geodesic curvature of a curve on a surface when the latter is subjected to point transformations. Generalizations of these theorems are sought and are introduced by the following paragraph: "Die obigen Sätze erfahren eine Erweiterung und finden zugleich ihren natürlichen Abschluss beim Uebergange zu Transformationen, welche aus der Anwendung des Lie'schen Begriffes der Berührungstransformationen auf die Flächentheorie hervorgehen. Wir können jedem Linienelement auf einer gegebenen Fläche ein bestimmtes Linienelement auf einer anderen gegebenen Fläche so zuordnen, dass je zwei, im Sinne des Herrn Lie vereinigt liegenden Elementen wieder zwei vereinigt liegende Elemente entsprechen. Dann wird jede auf der ersten Fläche gezogene Curve sich in eine Curve (in besonderen Fällen auch in einen Punkt) der zweiten Fläche verwandeln und zwei sich berührende Curven werden im Allgemeinen in zwei sich ebenfalls berührende Curven übergehen.* Die punktweise Transformation einer Fläche ist hierin als besonderer Fall enthalten."

This is correct if the footnote and last sentence be omitted. These call for modification in order to correct misappre-

* "Derartige Transformationen von Flächen sind meines Wissens noch nicht untersucht worden. Sie lassen sich, wie leicht zu sehen ist, den räumlichen Berührungstransformationen keineswegs unterordnen."

hensions that the reader of the paper referred to may carry away with him. These transformations have been studied; they fall under Lie's categories of contact transformations; and for space the transformations in question are nothing but point transformations.

There are two cases to consider: 1° when the given surface is transformed into itself; 2° when it is transformed into another surface. In each case, by the geometrical definition of the transformations, two associated lineal elements are to be transformed into two associated lineal elements, that is, element association is to be transformed into element association.*

In the first of the above two cases there are two classes of transformations possible: (*a*) those transformations of the two dimensional manifoldness which transform lineal element association into lineal element association, these are the proper contact transformations of the surface; (*b*) those transformations of space which leave the surface invariant and at the same time transform lineal element association into lineal element association.

In the second case considered above there is but one category of transformations to study, namely, (*c*) those transformations of space which transform associations of lineal elements in space into lineal element associations.

Of these three classes of transformations those of the first class (*a*) are proper contact transformations of a two-dimensional manifoldness; and the transformations (*b*) are a particular type of the transformations (*c*). This third category of transformations has been studied by Lie and the precise nature of its transformations determined.

A lineal element of the plane is the ensemble of a point and a line through the point. Two elements infinitely near to each other are said to be associated when the point of the first lies upon the line of the second to terms of the second order. A family of ∞^1 lineal elements is said to form an element association when every element is associated with all the infinitely neighboring elements of the family. An element association of lineal elements in the plane consists of the ∞^1 lineal elements of a point or of those of a curve. Contact transformations of the plane are those transformations of lineal elements which change element

* Those who are not wholly familiar with these notions will find them explained in a review of Lie's Geometry of Contact Transformations which appeared in the BULLETIN, 2d series, volume 3, number 9, June, 1897, pp. 321–350. Proofs of several theorems quoted later will be found in the fifth section of the tenth chapter of the above work.

association into element association; those which transform point into curve are proper contact transformations; those which transform point into point are the so-called extended point transformations.

The notion lineal element in the plane has two correspondents in ordinary space: 1° the aggregate of a point and a line through it; 2° the ensemble of a point and a plane through it, the latter being called a surface element. Two lineal elements are said to be associated in space in the same sense as in the plane. Two infinitely neighboring surface elements are said to be associated if the point of one lies in the plane of the other to terms of the second order. Element associations of lineal elements and element associations of surface elements are defined in an obvious manner analogous to the lineal element associations of the plane. Similarly there are transformations of lineal elements of space which transform lineal element association into lineal element association, and transformations of surface elements which transform surface element associations into surface element associations.

Now it is a curious fact that the direct extension of the notion lineal element association to space avails but little, while the modified extension to that of surface element association proves to be most fruitful. Lie recognized very early in his geometrical work that transformations of surface element associations into such include transformations which are not point transformations (since there are five varieties of surface element associations, namely the ∞^3 elements of a point, the ∞^2 of a curve, the ∞^2 of a surface, the ∞^1 of an element band along a curve, and finally the ∞^1 of an element cone), namely the so-called proper contact transformations of space; while those transformations which change lineal element association into lineal element association are always extended point transformations. Then the above corrections are completely borne out if it be further observed that of these lineal element associations in space there are three varieties: 1° the ∞^1 elements of a curve; 2° the ∞^1 elements of an element cone; 3° the ∞^2 elements of a point.

PRINCETON, N. J.,
21 *March*, 1898.

FOCAL PROPERTIES OF SURFACES OF THE SECOND ORDER.

Die Focaleigenschaften der Flächen zweiter Ordnung. Ein neues Kapitel zu den Lehrbüchern der analytischen Geometrie des Raumes. Von Professor Dr. OTTO STAUDE. Leipzig, Teubner, 1896. 8vo, viii + 186 pp.

THIS is a text-book which has grown from four articles by the author, published in: *Berichte der K. Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, 1882, p. 5, Mar. 3; 1895, p. 483, July 1; *Mathematische Annalen*, vols. 20 and 27.

The focal properties of the conic in the plane are so familiar that it is natural to enquire if there be not their counterpart in space. Investigations in this line for certain particular theorems were published by: Dupin (1813), Développement de géométrie; Steiner (1826), *Crelle* 1; Jacobi (1834), *Crelle* 12; Chasles (1835), *Mémoires de l'Institut*, and (1837), Notes; MacCullagh (1836), *Proceedings of the Irish Academy*; Lamé (1837), *Liouville* 2; Salmon (1842); Amiot (1843), *Liouville* 7, and (1845), 10; Plücker (1846), System der analytischen Geometrie des Raumes; Liouville (1847), *Liouville* 12; Townsend (1848), *Cambridge and Dublin Mathematical Journal* 3. This list of names makes it all the more strange that these masters of geometric analysis did not carry through the reasoning and find the focal properties. It was reserved for the author to simplify much that had gone before and to develop in a simple way the general theory by the use of the now common idea of a movable coördinate triheder. This development is also particularly valuable in that it uses throughout either the elliptic or parabolic coördinates, giving a most excellent presentation of these systems. And not the least important feature of the book is the care with which these exceptional cases are treated as part of the organic whole.

The first step in the development consists in using the equation to the cone from any point in space to each of the focal conics of a confocal system. This is found to be as follows: λ, μ, ν being the parameters of the three surfaces of the system through the point P; ξ, η, ζ, coordinates of any point referred to the three normals at P to these surfaces as axes; β, γ, the parameters corresponding to the two focal conics; then the equations of the cones from $P(\lambda, \mu, \nu)$ to the two focal conics are derived in the form (either for a central or a parabolic system):

$$(\gamma-\lambda)(\gamma-\mu)(\gamma-\nu)\left(\frac{\xi^2}{\gamma-\lambda}+\frac{\eta^2}{\gamma-\mu}+\frac{\zeta^2}{\gamma-\nu}\right)=0,$$

$$(\beta-\lambda)(\beta-\mu)(\beta-\nu)\left(\frac{\xi^2}{\beta-\lambda}+\frac{\eta^2}{\beta-\mu}+\frac{\zeta^2}{\beta-\nu}\right)=0.$$

The lines common to these cones are called the *focal lines* of P. There are four real lines, and, in general, they are distinct. If the point P be on the other focal conic the cone is right circular. Thence Professor Staude deduces many theorems, of which some had been known previously but not included in the general theory.

Possibly the most interesting, although not the most general of the focal properties which Professor Staude deduces, are the following. (The "focal distance" from any point to the focus of the one of a pair of focal conics is defined as the shortest distance on the broken line to a point on that conic, and thence to the adjacent focus).

In an ellipsoid, for any point on the surface, the sum of the focal distances to a focus of the focal ellipse and to the opposite focus of the focal hyperbola is a constant. And the normal to the surface at the point bisects the angle of the two lines. This theorem is the basis of the well-known wire model, No. 110 of the Brill collection, which represents the two focal curves of the system by wires rigidly connected, and which has a string fastened so as to represent the sum of the focal distances of the theorem.

In an hyperboloid of one sheet, for any point on the surface, the difference between the focal distances to a focus of the focal hyperbola and to the adjacent focus of the focal ellipse is a constant. And the normal to the surface at the point bisects the exterior angle of the two lines.

In an hyperboloid of two sheets, for any point on the surface, the difference between the focal distances to the two foci of a focal conic is a constant. And the normal to the surface at the point bisects the exterior angle of the two lines. It is easy to see how a string can be adjusted on the Brill model No. 110 to give the generation of the two hyperboloids.

For the confocal system of hyperboloids many new properties are developed. The property of the focal paraboloids corresponding to Dupin's focal properties is derived. (The plane having the properties of the directrix is called the principal directrix plane.)

In an elliptic paraboloid, for any point on the surface, the focal distance to the focus of the inner focal parabola is

equal to the normal upon the corresponding principal directrix plane. And the normal to the surface at the point bisects the exterior angle of the two lines.

In an hyperbolic paraboloid, for any point on the surface, the difference between the focal distances to the foci of the two focal parabolas is a constant. And the normal to the surface at the point bisects the exterior angle of the two lines.

The reader of this book will hope with Professor Staude "that the focal properties of the conicoid will attain the same recognition that the corresponding properties in the plane have long since enjoyed."

H. D. THOMPSON.

PRINCETON,
February 25, 1898.

NOTE ON THE THEORY OF CONTINUOUS GROUPS.

IN a "Note on the fundamental theorems of Lie's theory of continuous groups," contributed to the BULLETIN in November, 1897, by Dr. Lovett, attention is drawn to an error or misapprehension in a paper which I had the honor of contributing to the *Proceedings of the London Mathematical Society* (vol. 23, p. 381–390).

The theorem to which objection is taken is: "If $x_1', \cdots, x_n'$ is a point obtained from the point $x_1, \cdots, x_n$ by the operation

$$1 + X + \frac{X^2}{2!} + \cdots;$$

and $x_1'', \cdots, x_n''$ is a point obtained from the point $x_1', \cdots, x_n'$ by the operation

$$1 + Y' + \frac{Y'^2}{2!} + \cdots,$$

where $$X \equiv \lambda_1 X_1 + \cdots + \lambda_r X_r,$$

and $$Y \equiv \mu_1 X_1 + \cdots + \mu_r X_r,$$

X_k denoting the linear operator

$$\sum_{i=1}^{i=n} \xi_{ki}(x_1, \cdots, x_n) \frac{\partial}{\partial x_i};$$

then $x_1'', \cdots, x_n''$ can be directly derived from the point $x_1, \cdots, x_n$ by the operation

$$1 + Z + \frac{Z^2}{2!} + \cdots,$$

where $$Z \equiv \nu_1 X_1 + \cdots + \nu_r X_r$$

provided that, for all values of k, j

$$X_k X_j - X_j X_k \equiv \sum_{s=1}^{s=r} C_{kjs} X_s$$

where λ, μ, ν, and C are any functions of the variables."

The ground of the objection is that I implicitly assume a theorem which Lie has only proved when C_{kjs} is a constant. I wish to draw attention to the fact that in my proof I merely assume the obvious theorem

$$\underset{n=\infty}{Limit} \left(1 + \frac{Y}{n}\right)^n = 1 + Y + \frac{Y^2}{2!} + \cdots$$

where Y is any operator which obeys the distributive law. Lie's coustants C_{kjs} cannot, of course, appear in a simple group; and the fact that $Y \equiv \mu_1 X_1 + \cdots + \mu_r X_r$ cannot alter the fact that $1 + Y + \frac{Y^2}{2!} + \cdots$ is a simple group, though it can represent an infinity of simple groups by varying the forms of the functions $\mu_1, \cdots, \mu_r$.

It is not asserted that, if

$$X_k X_j - X_j X_k \equiv \sum_{s=1}^{s=r} C_{kjs} X_s,$$

then the operators $X_1, \cdots, X_r$ generate a *finite* continuous group but they do generate a continuous group. [I fear this is not clearly expressed in my paper.] Finally, there is not, so far as I can see, any discrepancy between the result which I have given and Lie's theorems; the latter are only concerned with *finite* continuous groups, though the reasoning by which they are established would, I think, hold good equally if his parameters $a_1, \cdots, a_r$ instead of being taken as constants were taken as arbitrary functions of $x_1, \cdots, x_n$, and the infinitesimal transformations formed by considering the effect of a small change in the form of any one of the functions. The conclusions would then, however, be nugatory unless for intransitive groups for which $r > n$.

J. E. CAMPBELL.

HERTFORD COLLEGE, OXFORD.

NOTES.

THE Fifth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the Institute of Technology, Boston, Mass., on Friday and Saturday, August 19th and 20th, thus immediately preceding the meeting of the American Association for the Advancement of Science. A colloquium will be held in connection with the Meeting. This will be conducted by Professor W. F. Osgood, who will give a course of lectures on some of the problems and methods of the modern theory of functions, and by Professor A. G. Webster who will treat certain portions of mathematical physics. A circular giving a more definite announcement in regard to the colloquium and the regular sessions will shortly be issued by the Secretary.

THE mathematical prize of the PRINCE JABLONOWSKI SOCIETY for 1897 was not awarded, no papers having been received. The prize subject proposed for 1898 was reported in the BULLETIN, vol. 2 (1896), p. 118. For 1901 the Society offers a prize of 1,000 marks for the best treatment of the following problem:

"*To complete in some essential feature the theory of the quadratic differential forms.*"

"The theory of the quadratic differential forms, which was first exploited by Riemann and was especially advanced by Christoffel and Lipschitz, has acquired a prominent significance through the modern investigations in geometry, dynamics, and the theory of transformation groups, and every advance in the pure theory would react beneficially in these various directions. Desiring that the theory of the quadratic differential forms should be completed in some essential feature, the Society invites the attention of the candidates especially to the question, raised by Lie's researches, of the nature and properties of the forms which admit of continuous groups of transformations. For the special case $n = 3$ Bianchi has recently obtained valuable results; it is to be hoped that the criteria for deciding whether a given form belongs to a preassigned type may be obtained in invariant form, and that the study of the geometries of the spaces in question may prove profitable.

Manuscript must be in the hands of the Secretary of the Society by November 30, 1901. Details as to anonymity, languages permissible, etc., may be obtained from the Secretary (for 1898, Professor A. Leskien, Leipzig, Stephanstrasse, 10).

THE philosophical faculty of the University of Göttingen announces, as the subject of the BENEKE PRIZE of 3400 marks (second prize, 680 marks) for the year 1901 the following:

"The *principle of continuity*, or, more specifically, *the possibility* of *representation by means of functions everywhere differentiable* was long regarded as the universal basis for the mathematical treatment of natural phenomena. This principle was introduced by the founders of the differential and integral calculus as self-evident; but the progress of mathematical investigation has shown more and more that this position involves a very large number of tacit assumptions which are by no means an essential consequence of the imperfection of our sense perception. Moreover, the principle in question is in contradiction to the accepted view of the molecular constitution of material bodies. The faculty desires a treatment, of genuine scientific interest, which shall present the questions involved in a generally comprehensible form and shall submit to thorough investigation the admissibility and the advantage of the prevailing hypothesis. The treatment may be developed on the mathematical, the philosophical, or the psychological side; historical studies are acceptable, but are not demanded."

Manuscripts, in one of the modern languages, must be sent in by August 31, 1900. The prizes will be awarded on March 11, 1901.

A CONGRESS of the mathematical sciences is one of twelve congresses of arts and sciences to be held during the Paris Exposition of 1900. M. Gariel, of the University of Paris, and secretary of the French Association for the Advancement of Science, has charge of the organization of the congresses.

PROFESSOR JAMES E. KEELER, director of the Allegheny Observatory and editor of the *Astrophysical Journal*, has accepted the directorship of the Lick Observatory.

PROFESSOR CHARLES EMILE PICARD has been recently elected a corresponding member of the Berlin Academy of Sciences.

AT the recent annual meeting of the New York Academy of Sciences, Dr. George W. Hill was elected honorary member.

PROFESSOR JAMES MCMAHON, Cornell University, has been elected general secretary of the American Association for the Advancement of Science.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CAYLEY (A.). The collected mathematical papers. Vol. XIV: Index to the 13 volumes. Cambridge, University Press, 1898. 4to. 5 and 141 pp. $2.50

DIXON (E. T.). A paper on the foundations of projective geometry. Cambridge, 1898. 8vo. 84 pp. 2s.

DOEHLMANN (K.). Projective Geometrie in synthetischer Behandlung. (Sammlung Goeschen, No. 72.) Leipzig, Goeschen, 1898. 12mo. 162 pp. Cloth. Mk. 0.80

DONATI (L.). Appunti di analisi vettoriale. Bologna, 1897. 4to. 26 pp. Fr. 2.50

GERBALDI (F.). Sul gruppo semplice di 360 collineazioni piane. (*Rendiconti del Circolo Matematico di Palermo*, Vol. XII, 1898.) 8vo. 72 pp.

HOCHHEIM (A.). Problèmes de géométrie analytique à deux dimensions. Ouvrage traduit de l'allemand sur la deuxième édition par L. Isely-Delisle. Fascicule I: Point, droite et cercle. Paris, Gauthier-Villars, 1897. 8vo. Fr. 2.00

HUTCHINSON (J. I.). On the reduction of hyperelliptic functions ($p = 2$) to elliptic functions by a transformation of the second degree. [Diss.]. Göttingen, Kaestner. 1897. 8vo. 40 pp.

ISELY-DELISLE (L.). See HOCHHEIM (A.).

IERMAKOV (W. P.). The theory of Abelian functions and Riemann surfaces. Kiev, 1897. 8vo. 121 pp. (Russian.) Mk. 5.00

JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Herausgegeben von E. Lampe. Vol. 26: Jahrgang 1895. (In 3 Heften.) Heft 3. Berlin, Reimer, 1898. 8vo. 62 pp. and pp. 785–1173. Mk. 11.40

KILLING (W.). Einführung in die Grundlagen der Geometrie. Band II: Congruenz und Messung; Abschluss der projectiven Geometrie; Grundbegriffe und Grundsätze der Geometrie; Anwendung der Transformationsgruppen. Paderborn, 1898. 8vo. 6 and 361 pp. Mk. 7.00

———. Karl Weierstrass. Münster, 1898. 8vo. 24 pp. Mk. 0.60

KLEIN (F.). Sur la géométrie dite non euclidienne. Traduit par L. Laugel. (*Faculté de Toulouse*, Vol. XI.) 4to. 62 pp.

LAMPE (E.). See JAHRBUCH.

LAUGEL (L.). See KLEIN (F.).

MININ (A. P.). Collection of problems in analytic geometry. (Russian.) Moscow, 1897. 8vo. 62 pp. Mk. 2.50

NÉDÉLEC (G.). Le calcul vectoriel et ses applications en géométrie et en mécanique. (En 2 volumes.) Volume I. Paris, Gauthier-Villars, 1897. 8vo. 10 and 246 pp. Fr. 6.00

RÉMOND (A.). Exercices élémentaires de géométrie analytique à deux et à trois dimensions, avec un exposé des méthodes de résolution, suivis des énoncés des problèmes donnés pour les compositions d'admission aux écoles polytechnique, normale, centrale, navale, au concours général, à l'agrégation. Partie I: Géométrie à deux dimensions. Paris, Gauthier-Villars, 1898. 8vo. 8 and 336 pp. Fr. 7.00

———. Partie II: Géométrie à trois dimensions; problèmes généraux; énoncés. 2e édition. Paris, Gauthier-Villars, 1898. 8vo. Fr. 7.00

ROSSEVICH (I. N.). See SERRET (J. A.).

SCHRÖTER (H.). See STEINER (J.).

SERRET (J. A.). Higher Algebra. Russian translation by I. N. Rossevich. Parts I. and II. St. Petersburg, 1897. 8vo. 615 pp. Mk. 10.00

STEINER (J.). Vorlesungen über synthetische Geometrie. Theil II: Die Theorie der Kegelschnitte, gestützt auf projective Eigenschaften. Auf Grund von Universitätsvorträgen und mit Benutzung hinterlassener Manuscripte J. Steiners bearbeitet von H. Schröter. 3te Auflage, durchgesehen von R. Sturm. Leipzig, Teubner, 1898. 8vo. 17 and 537 pp. Mk. 14.00

STURM (R.). See STEINER (J.).

VALLÉE POUSSIN (J. DE LA). Recherches analytiques sur la théorie des nombres. Paris, Hermann, 1897. 8vo. Fr. 10.00

WEBER (H.). Lehrbuch der Algebra. 2te Auflage. Vol. I. Braunschweig, Vieweg, 1898. 15 and 703 pp. Mk. 10.00

WHITEHEAD (A. N.). A treatise on universal algebra, with some applications. Vol. I. Cambridge, UniversityPress (New York, Macmillan Co.), 1898. 8vo. 26 and 586 pp. Cloth. $7.00

II. ELEMENTARY MATHEMATICS.

ALEKSANDROV (I.). Methods for the solution of geometric problems and collection of geometric problems with complete brief solutions. 6th edition. (Russian.) Moscow, 1897. 8vo. 175 pp. Mk. 3.50

BASSANI (A.). See LAZZARI (G.).

BUCK (R. C.). A manual of trigonometry. Philadelphia, Lippincott, 1897. 12mo. Cloth. $1.50

CRAWLEY (E. S.). Elements of plane and spherical trigonometry. A text-book for colleges and schools. 2d edition, revised and enlarged. Philadelphia, Crawley, 1898. 12mo. 175 pp. Cloth. $1.00

COMBEROUSSE (C. DE). See ROUCHÉ (E.).

DUCATEL (A.). See PICHOT (J.).

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 54ste Auflage. Halle, 1898. 8vo. 166 and 35 pp. Cloth. Mk. 2.50

GIKA (D.) and MUROMTSEV (A.). Geometrical problems. Part I: 1773 problems in plane geometry. 5th edition. (Russian.) Moscow, 1897. 8vo. 190 pp. Mk. 3.00

GOODWIN (H. B.). Elementary plane trigonometry. Abridged from the work on plane and spherical trigonometry used as a text-book in the Royal Navy. London, Longmans, 1898. 8vo. 208 pp. 5s.

HADAMARD (J.). Leçons de géométrie élémentaire (géométrie plane). (Cours complet de mathématiques élémentaires, publié sous la direction de G. Darboux.) Paris, Colin, 1898. 8vo. 16 and 308 pp. Fr. 6.00

HALL (H. S.) and KNIGHT (S. R.). Key to elementary trigonometry. New York, Macmillan, 1898. 12mo. Cloth. $2.25

KNIGHT (S. R.). See HALL (H. S.).

KORYTIN (S. P.). Review of the educational literature on arithmetic and geometry. (Russian.) St. Petersburg, 1897. 8vo. 155 pp. Mk. 4.00

LAZZARI (G.) e BASSANI (A.). Elementi di geometria. 2a edizione. Livorno, 1898. 8vo. 396 pp. Fr. 5.00

MOEHL (E. G.). Ready reckoner in German and English. Mathematical universal manual, with directions, examples and complete computation tables, for the practical use of technologists, engineers, builders and merchants. Milwaukee, Moehl, 1898. 16mo. 143 pp. Cloth. $0.75

MUROMTSEV (A.). See GIKA (D.).

PICHOT (J.). Eléments de trigonométrie rectiligne, à l'usage de la classe de mathématiques élémentaires, des candidats aux baccalauréats

classique et moderne, à l'Ecole de St.-Cyr et à l'Institut agronomique. Nouvelle édition, entièrement revue et mise en harmonie avec les dernières programmes par A. Ducatel. Paris, Hachette, 1898. 8vo. 223 pp. Fr. 3.50

PRZHEVALSKY (E.). Collection of problems in plane trigonometry. 5th edition. (Russian.) Moscow, 1897. 8vo. 248 pp. Mk. 5.50

SAILER (E.). Die Aufgaben aus der Elementarmathematik, welche bei der Prüfung für das Lehramt der Mathematik und Physik an den k. bayersichen humanistischen und technischen Unterrichtsanstalten in den Jahren 1873 bis 1893 gestellt wurden, bearbeitet von E. Sailer. München, Ackermann, 1898. 8vo. 3 and 167 pp. Mk. 3.80

SCHUBERT (H.). Arithmetik und Algebra. 2te Auflage. (Sammlung Goeschen, No. 47.) Leipzig, Goeschen, 1898. 12mo. 171 pp. Cloth. Mk. 0.80

SCOTT (E. E.). Short table of logarithms and anti-logarithms to ten places. London, 1898. 8vo. Cloth. 10s.

SHAPOSHNIKOV (N. A.) and VALTSOV (N. K.). Collection of problems in algebra. Part II. 5th edition. (Russian.) Moscow. 1897. 8vo. 194 pp. Mk. 3.00.

SICKENBERGER (A.). Uebungsbuch zur Algebra. Abteilung II: Dritte Stufe der Rechnungsarten, quadratische Gleichungen, Reihen. 2te Auflage. München, Ackermann, 1898. 8vo. 3 and 128 pp. Boards. Mk. 2.00

VALTSOV (N. K.). See SHAPOSHNIKOV (N. A.).

III. APPLIED MATHEMATICS.

COTTIER (J.). The equations of hydrodynamics in a form suitable for application to problems connected with the movements of the earth's atmosphere. Prepared at the request of W. L. Moore. (U. S. Department of Agriculture, Weather Bureau, No. 130.) Washington, 1897. 4to. 8 pp.

DIESENER (H.). Praktische Unterrichtsbücher für Bautechniker. Vol. I: Darstellende Geometrie; das geometrische Zeichnen; Projektionslehre; die Lehre vom Steinschnitt; die Schattenkonstruktionen; die Perspective und die Farbenlehre. Leicht fasslich dargestellt für Selbstunterricht und Schulgebrauch. 4te Auflage. Halle, Hofstetter, 1898. 8vo. 6 and 149 pp. Mk. 4.00

FEDOROV (P. A.). See POINSOT (L.).

GALLENMÜLLER (J.). Elemente der mathematischen Geographie und Astronomie. 3te Auflage. Regensburg, 1898. 8vo. 180 pages, 1 map. Mk. 1.70

GANORA (G.). Elementi di disegno ornamentale geometrico. Parte I: Costruzioni grafiche nel piano. Torino, 1898. 8vo. 140 pp. Fr. 4.00

GÜNTHER (L.). See KEPLER (J.).

HENTZE (W.). Analytische Berechnung elektrischer Leitungen. Berlin, Springer, 1898. 8vo. 5 and 81 pp. Cloth. Mk. 3.00

HOFF (J. H. van't). Vorlesungen über theoretische und physikalische Chemie. Heft I: Die chemische Dynamik. Braunschweig, Vieweg, 1898. 8vo. 11 and 252 pp. Mk. 6.00

HUMMEL (L.). See REBBER (W.).

IEVNEVICH (I. A.). Text-book of applied mechanics. 2nd edition. (Russian.) St. Petersburg, 1897. 8vo. 390 pp., with plates. Mk. 11.00

J. (F.). Éléments de géométrie descriptive, avec de nombreux exercices. 6e édition. Paris, Poussielgue, 1898. 16mo. 462 pp.

———. Éléments de mécanique, avec de nombreux exercices. 4e édi-

tion, répondant aux programmes des classes de mathématiques élémentaires et de première-sciences. Paris, Poussielgue, 1897. 16mo. 8 and 319 pp.

LAURENT (H.). Théorie des opérations financières. Paris, Gauthier-Villars [1898].

LE DANTEC. La physique de la lumière. Fascicule 2: Analyse physique de la polarisation chromatique basée sur la mécanique de l'élasticité d'un ellipsoïde. Paris, Fritsch, 1898. 8vo. 96 pp. Fr. 3.00

———. La physique de la lumière. Fascicule 4: Analyse critique de la théorie des ondes, ou mes interviews avec les savants au sujet de la lumière et de l'électricité. Paris, Fritch, 1898. 8vo. 236 pp. Fr. 3.00

LÜLING (E.). Mathematische Tafeln für Markscheider und Bergingenieure, sowie zum Gebrauche für Bergschulen. 4te erweiterte Auflage. Berlin, 1898. 8vo. 41 and 64 pp. Cloth. Mk. 6.00

MAISS (E.). Aufgaben über Wärme einschliesslich der mechanischen Wärmetheorie und der kinetischen Theorie der Gase. Für Studierende an Mittel- und Gewerbeschulen, zum Selbststudium für angehende Techniker, Physiker und andere. Wien, Pichler. 1898. 8vo. 5 and 118 pp. Mk. 2.40

MARTIN (E.) et PERNOT (F.). Cours de géométrie descriptive. (En 2 volumes.) Vol. II: Surfaces de révolution; intersection des surfaces; surfaces du 2e ordre. Paris, 1898. 8vo. 500 pp. Fr. 10 00

MESHCHERSKY (J.). Investigations in dynamics. (Russian.) St. Petersburg, 1897. 8vo. 468 pp. Mk. 6.00

MILLAR (W. J.). See RANKINE (W. J. M.).

MOEBIUS (A. F.). Astronomie; Grösse, Bewegung und Entfernung der Himmelskörper. 9te Auflage, von W. F. Wislicenus. (Sammlung Goeschen, No. 11.) Leipzig, Goeschen, 1898. 12mo. 164 pp. Cloth. Mk. 0.80

PERNOT (F.). See MARTIN (E.).

PIGNANT (P.). Eléments de géométrie descriptive. Dijon, Jacquot et Floret [1898]. 4to. 55 pp.

POINSOT (L.). Elements of Statics. Translated into Russian by P. A. Fedorov. St. Petersburg, 1897. 8vo. 218 pp. Mk. 3.50

RANKINE (W. J. M.). Manual of applied mechanics. 15th edition revised by W. J. Millar. London, Griffin, 1898. 8vo. 696 pp. Cloth. 12s. 6d.

REBBER (W.). Die Festigkeitslehre und ihre Anwendung auf den Maschinenbau. Elementar behandelt zum Gebrauche für Studierende und in der Praxis. 3te Auflage, herausgegeben von L. Hummel. Mitweida, Polytechnische Buchhandlung, 1898. 8vo. 16 and 476 pp. Mk. 10.50.

REY-PAILHADE (J. DE). L'extension du système décimal aux mesures du temps et des angles. Théorie; applications scientifiques et industrielles. Paris, Gauthier-Villars, 1897. 8vo. 1 plate. Fr. 2.00.

SARAZIN (C.). Cours d'électricité; théorie et pratique. Paris, Bernard, 1898. 8vo. 6 and 632 pp. Fr. 16.00.

TOLOMEI (G.). Sopra la forma di equilibrio di una massa fluida animata da un movimento di rotazione. Firenze, Fiorentino, 1898. 8vo. 59 pp.

VILLIÉ (E.). Compositions d'analyse; cinématique, mécanique et astronomie; données depuis 1889 á la Sorbonne pour la licence ès sciences. Enoncés et solutions. Partie III. Paris, Gauthier-Villars, 1898. 8vo. 10 and 300 pp.

WILLSON (F. N.). Theoretical and practical graphics. Princeton, 1897. 4to. 8 and 264 pp.

WISLICENUS (W. F.). See MOEBIUS (A. F.).